Professor Carl Cowen **Math 53000** **Spring 14**

PROBLEMS

Note: The dates in this problem list indicate the dates by which you should have answered/thought about these questions.

Note: $*$ indicates a problem that will be graded. $**$ indicates a problem that will be handed in separately, graded, and has opportunity to be corrected (once).

January 21 (NOTE: There will be a quiz over this material the last 15 minutes of class on January 21.)

1. Let $z = 4 - 5i$.
Find: (a) $\mathrm{Re}(z)$ (b) $\mathrm{Im}(z)$ (c) $|z|$ (d) $\bar{z}$.

2. Compute:

(a) $(3+2i)(2-i) + i(-2+i)$ (b) $(2-3i)^2(4+2i)$

(c) $(2-i)^2 + (1+3i)^2$ (d) $\left(\overline{(2-i)}\right)^2 + \left(\overline{(1+3i)}\right)^2$ *see (c)*

(e) $\dfrac{1}{3+4i}$ (f) $\dfrac{4-2i}{1+i}$

(g) $\dfrac{2+3i}{(2-i)^2} + \dfrac{i}{1+i}$ (h) $\left|\dfrac{1+3i}{(2-i)}\right|$.

3. Find all of the complex numbers that deserve to be called $\sqrt{5-2i}$. How many are there?

4. Find all (3) roots of the equation $z^3 - 3z^2 + 7z - 5 = 0$.

5. Prove that if p is a polynomial with real coefficients and r is a root of p, then $\overline{r}$ is also a root of p.

6. Suppose that $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, where $a_n a_0 \neq 0$, is a polynomial with roots $r_1, r_2, \cdots, r_{n-1}, r_n$. Let $q(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$. Find the roots of q and prove that your answer is correct.

7. Use the definition of the complex numbers as ordered pairs of real numbers, $z = (r, s)$ and plus, $+$, and times, $*$, defined by

$$z_1 + z_2 = (r_1, s_1) + (r_2, s_2) \quad \text{and} \quad z_1 * z_2 = (r_1, s_1) * (r_2, s_2) = (r_1 r_2 - s_1 s_2, r_1 s_2 + r_2 s_1)$$

and the usual properties of real number arithmetic, prove the associative law for complex multiplication.

8. Prove that $-|z_1 - z_2| \leq |z_1| - |z_2| \leq |z_1 - z_2|$.

9. Prove that if $|z_1| = 1$ and $z_1 \neq z_2$, then $\left|\dfrac{z_1 - z_2}{1 - \overline{z_2} z_1}\right| = 1$.

10. Prove that if $|z_1| < 1$ and $|z_2| < 1$, then $\left|\dfrac{z_1 - z_2}{1 - \overline{z_2} z_1}\right| < 1$.

11. Show that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$. If we think of z_1 and z_2 as vectors in the plane, what does this mean geometrically?

12. Find all third and fourth roots of i.

13. Prove that for each positive integer $n \geq 2$, the sum of the n^{th} roots of unity is 0.

14. Let $w \neq 1$ be an n^{th} root of 1. Calculate:
(a) $1 + 2w + 3w^2 + \cdots + nw^{n-1}$
(b) $1 + 4w + 9w^2 + \cdots + n^2 w^{n-1}$
(Hint: Multiply by $1 - w$.)

15. (a) Show that if t is a real number, then $\left|\dfrac{1+it}{1-it}\right| = 1$.

(b) Show that if $|z| = 1$ and $z \neq -1$, then there is a real number t so that $z = \dfrac{1+it}{1-it}$.

16. If z is in $\partial\mathbb{D}$, that is, $|z| = 1$, then $f(z) = \dfrac{2z+i}{(1-2i)z-3}$ lies on a circle in the complex plane, $\mathbb{C}$. Find the center and the radius of this circle.

17. Show that the set $\Gamma = \{z \in \mathbb{C} : |z - 3 + 4i| + |z + 2| = 7\}$ is non-empty and describe the set geometrically.

18. Give a geometric description of the set $\{z \in \mathbb{C} : \mathrm{Re}(2 + 3i)z \geq 1\}$.

Stereographic projection/extended complex plane/Riemann sphere

The *extended complex plane* or *Riemann sphere* is the set $\mathbb{C} \cup \{\infty\}$, that is, the complex plane together with the 'point at infinity'.

A concrete construction of this object is given by *stereographic projection* of the unit sphere in $\mathbb{R}^3$, that is, $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ onto the complex plane $\mathbb{C}$, thought of as the plane $\{(x, y, 0) : x, y \in \mathbb{R}\}$ where $z = x + iy$, with the mapping of the plane onto the sphere, except the 'North pole', $(0, 0, 1)$, by taking the point $(x, y, 0)$ on the plane to the point of the sphere where the line, ℓ, in $\mathbb{R}^3$ through $(0, 0, 1)$ and $(x, y, 0)$ intersects the sphere. The function $\gamma(t) = (1 - t)(0, 0, 1) + t(x, y, 0) = (tx, ty, 1 - t)$ parametrizes the line ℓ, and the point at which it intersects the sphere is the point for which $(tx)^2 + (ty)^2 + (1 - t)^2 = 1$, or

$$1 = t^2(x^2 + y^2) + (1 - t)^2 = t^2|z|^2 + (1 - t)^2$$

which gives $t = 0$, the 'North pole', or $t = 2/(1 + |z|^2)$. That is,

$$z \mapsto \left(\frac{2\,\mathrm{Re}z}{|z|^2+1}, \frac{2\,\mathrm{Im}z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right) \quad \text{and} \quad (\alpha, \beta, \gamma) \mapsto \frac{\alpha + \beta i}{1 - \gamma}$$

for z in the complex plane and $(\alpha, \beta, \gamma) \neq (0, 0, 1)$ in the sphere. In the extended complex plane, the point ∞ corresponds to the 'North pole' in the sphere, so the extended complex plane is identified with the unit sphere in $\mathbb{R}^3$. The unit sphere is a compact subset of $\mathbb{R}^3$ and inherits a topology, that is a collection of open subsets, from $\mathbb{R}^3$ and it is not hard to see that the usual (Euclidean) topology on the complex plane thought of as identified with $\mathbb{R}^2$ and the topology on the sphere without the north pole are the same.

19. Which points in the plane correspond to $(0, 0, -1)$? to the equator of the sphere?, to a circle on the sphere that passes through $(0, 0, 1)$? What points in the sphere correspond to circles with center 0 in the plane? the half disk $\{z \in \mathbb{C} : |z| \leq 1 \text{ and } \mathrm{Im}(z) \geq 0\}$?

20. Which sequences $(z_n)_{n=1}^{\infty}$ in $\mathbb{C}$ correspond, under the stereographic projection, $z_n \leftrightarrow p_n$, to sequences $(p_n)_{n=1}^{\infty}$ in the unit sphere to sequences that converge to $(0, 0, 1)$? In other words if $z_n \leftrightarrow p_n$ find conditions so that the sequence (z_n) satisfies these conditions if and only if $\lim_{n\to\infty} p_n = (0, 0, 1)$.

January 30

$*$ **21.**
Let $\gamma(t) = (3+2i)t + (2-i)t^2$ for $-1 \le t \le 1$ and let $\Gamma = \{\gamma(t) : -1 \le t \le 1\}$ which is a smooth curve in $\mathbb{C}$.
(a) Find a parametrization of the line ℓ_1 tangent to the curve Γ at $\gamma(0)$.
(b) Find a parametrization of the line ℓ_2 tangent to the curve Γ at $\gamma(.5)$.
(c) Find the angles between the real axis and these tangent lines.

$**$ **22.**
Let G be an open, non-empty subset of $\mathbb{C}$, let γ be a differentiable function mapping the interval $[-1,1]$ into G, that is, the set $\Gamma = \{\gamma(t) : -1 \le t \le 1\}$ is a smooth curve in G.
(a) Find a parametrization of the line tangent to the curve Γ at $\gamma(0)$
(b) Find the angle between the real axis and this tangent line.
(c) Suppose h is a complex valued function that is differentiable in G. Then $h(\Gamma)$ is a smooth curve in $h(G)$. Find a parametrization of the line tangent to $h(\Gamma)$ at the point $h(\gamma(0))$.
(d) Find the angle between the real axis and the line tangent to $h(\Gamma)$ at the point $h(\gamma(0))$.

$*$ **23.**
For which real numbers a, b, c, and d is the function $u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$ harmonic on $\mathbb{C}$? For the cases in which u is harmonic, find a harmonic conjugate, v, of u.

24. Prove that if u is harmonic and v is a harmonic conjugate of u, then uv and $u^2 - v^2$ are both harmonic, but u^2 harmonic implies u is constant.

For problems 25 to 28, a, b, c, and d are complex numbers with $ad - bc \ne 0$ and φ is the linear fractional map $\varphi(z) = \dfrac{az+b}{cz+d}$. Let $\widehat{\mathbb{C}}$ denote the Riemann sphere and for $c = 0$, regard $\varphi(\infty)$ as ∞, and for $c \ne 0$, regard $\varphi(\infty)$ as a/c and $\varphi(-d/c)$ as ∞.

$*$ **25.**
Show that the linear fractional map φ is injective, that is, one-to-one, on $\widehat{\mathbb{C}}$.

26. Show that $\psi(z) = \dfrac{dz-b}{-cz+a}$ is the inverse of φ, that is, $\psi(\varphi(z)) = z$ and $\varphi(\psi(z)) = z$ for each z in $\widehat{\mathbb{C}}$. We will write φ^{-1} for ψ.

$*$ **27.**
Show that if z_1, z_2, and z_3 are three distinct points in $\widehat{\mathbb{C}}$, then there is a unique linear fractional map ζ so that $\zeta(z_1) = 0$, $\zeta(z_2) = 1$, and $\zeta(z_3) = \infty$. Conclude, by using $\eta^{-1} \circ \zeta$ where η is a linear fractional map that takes w_1, w_2, and w_3 onto 0, 1, and ∞, that for any distinct points z_1, z_2, and z_3, and distinct points w_1, w_2, and w_3, there is a unique linear fractional map φ with $\varphi(z_1) = w_1$, $\varphi(z_2) = w_2$, and $\varphi(z_3) = w_3$.

$*$ **28.**
Find a linear fractional map φ that takes the real axis onto itself, and the unit disk, $\mathbb{D} = \{z : |z| < 1\}$, onto the half-plane $\{w : \mathrm{Re}(w) > 2\}$.

February 6

$*$ **29.**
Show that every linear fractional map φ has one or two fixed points in $\widehat{\mathbb{C}}$ and that if there is only one fixed point, p, then $\varphi'(p) = 1$. (This is the condition for a 'fixed point of multiplicity two' (or more).)

30. Given four distinct points z_1, z_2, z_3, and z_4 in $\widehat{\mathbb{C}}$, their *cross ratio*, denoted $(z_1, z_2; z_3, z_4)$, is defined to be the image of z_4 under the linear fractional transformation that sends z_1, z_2, and z_3 respectively to ∞, 0, and 1. Prove that if φ is any linear fractional transformation, then $(\varphi(z_1), \varphi(z_2); \varphi(z_3), \varphi(z_4)) = (z_1, z_2; z_3, z_4)$.

$*$ **31.**
Let ψ be the linear fractional map such that $\psi(\infty) = 1$ and has i and $-i$ as fixed points. Find $\psi(\{z : \mathrm{Re}(z) > 0\})$ and also find $\psi(\mathbb{D})$, where $\mathbb{D}$ is the open unit disk.

$*$ **32.**
(a) Prove that $\varphi(z)$ is a linear fractional map of the half-plane $\{z : \mathrm{Im}(z) > 0\}$ onto itself if and only if there are a, b, c, and d real numbers with $ad - bc = 1$ such that $\varphi(z) = \dfrac{az + b}{cz + d}$. Conclude that the set of such linear fractional maps forms a group under composition.

(b) Show that the set of φ such that a, b, c, and d are integers with $ad - bc = 1$ form a subgroup of the group in part (a). This group is known as the *Modular Group* and is important in number theory, geometry, and automorphic forms.

$*$ **33.**
Prove that the linear fractional transformations mapping the disk $\mathbb{D}$ onto itself are those of the form $\varphi(z) = \lambda \dfrac{z - z_0}{1 - \overline{z_0} z}$ for some $|\lambda| = 1$ and $|z_0| < 1$.

34. Find the image under the linear fractional map $\varphi(z) = \dfrac{z - 1}{z + 1}$ of the intersection of $\mathbb{D}$ with $\{z : \mathrm{Im}(z) > 0\}$.

35. Find all linear fractional transformations that fix 1 and -1. Is this group abelian? Can you identify this group?

$**$ **36.**
Let $\exp(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

(a) Show the series for exp converges for every z in $\mathbb{C}$.

(b) Give a careful proof that $\exp(a+b) = \exp(a)\exp(b)$ for all complex numbers a and b.

February 13

$*$ **37.**
Prove: *If* $(b_k)_{k=1}^{\infty}$ *is a sequence of complex numbers with* $\lim_{k\to\infty} b_k = L \neq \infty$*, then for* $a_n = \frac{1}{n}(b_1 + b_2 + \cdots + b_n)$*, we have* $\lim_{n\to\infty} a_n = L$ *also.*

$*$ **38.**
Prove the *Limit Comparison Test*: *If* $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *are sequences of positive real numbers with* $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L$*, where* $0 < L < \infty$*, then either both series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *converge or both series diverge.* As a corollary, prove that if

$$\sum_{n=0}^{\infty} c_n(z-a)^n \quad \text{and} \quad \sum_{n=0}^{\infty} d_n(z-a)^n$$

are power series for which $|c_n|$ and $|d_n|$ satisfy the hypotheses of the limit comparison test, then the power series have the same radius of convergence.

$*$ **39.**
Prove that the power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ and $\displaystyle\sum_{n=0}^{\infty} n a_n z^{n-1}$ have the same radius of convergence.

40. Use 'long division' to divide 1 by $1 + z + z^2$ (*NOT* $z^2 + z + 1$!!) to get a power series for $\dfrac{1}{1+z+z^2}$. Then factor your answer, add the resulting series, and thereby convince yourself that your answer was correct!

41. Find formulas for the sums of the series:

(a) $\displaystyle\sum_{n=0}^{\infty}(n+1)z^n$ (c) $\displaystyle\sum_{n=1}^{\infty} n^2 z^n$

(b) $\displaystyle\sum_{n=1}^{\infty} n z^n$ (d) $\displaystyle\sum_{n=0}^{\infty}(2^{-n} + 3^{-n})z^n$

$*$ **42.**
In problem 36, we defined $\exp(z) = 1 + z + \dfrac{z^2}{2!} + \cdots$ which implies $\exp(0) = 1$. Suppose λ is a function defined near 1 with $\lambda(1) = 0$ and $\lambda(\exp(z)) = z$ for z near 0.

(a) Assuming λ has a power series expansion near 1, so that $\lambda(w) = c_1(w-1) + c_2(w-1)^2 + c_3(w-1)^3 + \cdots$, what must the coefficients c_j be for $\lambda(\exp(z)) = z$.
(b) What is the radius of convergence of the power series for λ.

$**$ **43.**
Suppose $0 < |s| < 1$ and $\varphi(z) = sz + a_2 z^2 + a_3 z^3 + \cdots$ where the power series for φ converges in a disk containing $\mathbb{D}$.

(a) Show that if $f(z) = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots$ then it is possible to find each of the coefficients of the composite function $g(z) = f(\varphi(z))$ in finite time (although usually the time increases as n increases).
(b) Find necessary conditions on λ and f for the functional equation $f(\varphi(z)) = \lambda f(z)$ to have a solution: Find a theorem *If* f *is a solution of* $f(\varphi(z)) = \lambda f(z)$*, then* $\cdots$.

February 20

$*$ **44.**
Show that there are two branches of the function $f(z) = \sqrt{z^2-1}$ that are defined and continuous on the set $\Omega = \mathbb{C}\setminus[-1,1] = \{z : |z+1|+|z-1| > 2\}$. Labeling f_+ the branch with value $f_+(2.6) = 2.4$ and f_- the branch with value $f_-(2.6) = -2.4$. Choose three positive real numbers, r, s, and t so that $z_1 = ri$, $z_2 = -s$, $z_3 = -ti$ are in Ω and compute the values of the branches $f_+(z_j)$ and $f_-(z_j)$ for $j = 1, 2,$ and 3.

$*$ **45.**
Find a suitable domain, Ω_0, (i.e. Ω_0 is an open, connected, non-empty set) on which the three branches of the function $g(z) = \sqrt[3]{z^3}$ are defined, continuous, and holomorphic, and describe the branches of g on this domain. If Ω_0 is not maximal, that is, if there is a domain $\Omega \neq \Omega_0$ such that $\Omega \supset \Omega_0$ and the three branches are holomorphic and single valued on Ω, then find a maximal domain.
(Hint: One way to do this is to realize that if $w = w(z)$ is a branch of g, then $w^3 = z^3$. Then, you can solve this equation for w in terms of z.)

$**$ **46.**
The function $B(z) = z\left(\dfrac{z^2-1/4}{1-z^2/4}\right)$ is a 3-to-1 mapping of the unit disk $\mathbb{D}$ onto itself. For example, we have $B(0) = B(1/2) = B(-1/2) = 0$, and no other points of $\mathbb{D}$ are mapped to 0. There are three branches of $B^{-1}\circ B$ defined near 0; one satisfies $g_1(0) = 0$, one satisfies $g_2(0) = 1/2$, and the other satisfies $g_3(0) = -1/2$. Describe the three branches of $B^{-1}\circ B$ and a domain $\Omega \subset \mathbb{D}$ so that all three branches are holomorphic and single valued in Ω and the closure of Ω contains $\mathbb{D}$.

$*$ **47.**
Compute $\int_\gamma f(z)\,dz$ for $f(z) = z^2$ and γ the curve defined by

$$\gamma(t) = \begin{cases} 1+(t-1)i & 0 \le t < 2 \\ (3-t)+i & 2 \le t < 4 \\ -1+(5-t)i & 4 \le t < 6 \\ (t-7)-i & 6 \le t \le 8 \end{cases}$$

48. (a) Compute $\int_\zeta g(z)\,dz$ for $g(z) = 1/z$ and ζ the unit circle parametrized by $\zeta(\theta) = e^{i\theta}$, for $0 \le \theta \le 2\pi$.
(b) Compute $\int_\zeta h(z)\,dz$ for $h(z) = 1/z^2$ and ζ the unit circle parametrized as above.

$*$ **49.**
Let f be holomorphic in the domain Ω. Suppose γ and ζ are smooth curves (i.e. γ, ζ, γ', and ζ' are continuous) in Ω such that $\gamma(t)$ is defined for $a \le t \le b$ and $\zeta(s)$ is defined for $c \le s \le d$ for real numbers a, b, c, and d with $\gamma(a) = \zeta(c)$ and $\gamma(b) = \zeta(d)$ and $\gamma'(t)$ and $\zeta'(s)$ are never 0. In addition, suppose that the numbers $\gamma(t)$ are distinct for $a \le t < b$, the numbers $\zeta(s)$ are distinct for $c \le s < d$ and the sets $\{\gamma(t) : a \le t \le b\}$ and $\{\zeta(t) : c \le t \le d\}$ are the same. Decide whether we always have

$$\int_\gamma f(z)\,dz = \int_\zeta f(z)\,dz$$

and justify your answer.

March 6

$*$ **50.**
Use the theorem "*If f is analytic in $G = \{z : |z-a| < R\}$, for a in $\mathbb{C}$ and $0 < R$, and γ is a piecewise C^1 closed curve in G, then $\int_\gamma f(z)\,dz = 0$.*" to prove the analogous result for H an open half plane in $\mathbb{C}$ (such as $H = \{z : \mathrm{Re}(z) > -3\}$).

$*$ **51.**
Evaluate the integrals $\displaystyle\int_0^\infty \cos(t^2)\,dt$ and $\displaystyle\int_0^\infty \sin(t^2)\,dt$ by integrating e^{-z^2} along a curve that follows the boundary of the region $\{z : |z| \le R \text{ and } 0 \le \arg(z) \le \pi/4\}$ in a counter-clockwise direction and then taking the limit as R goes to ∞. (These are called the Fresnel integrals after the French engineer who invented what are now called Fresnel lenses and needed such integrals in his study of optics.)

$*$ **52.**
Evaluate the following integrals

(a) $\displaystyle\int_\gamma \frac{e^{iz}}{z^2}\,dz \quad \text{for } \gamma(t) = e^{it}, \quad 0 \le t \le 2\pi$

(b) $\displaystyle\int_\gamma \frac{p(z)}{z-a}\,dz \quad \text{for } \gamma(t) = a + re^{it}, \quad 0 \le t \le 2\pi, \quad$ and p a polynomial

$*$ **53.**
Evaluate the integral

$$\int_\gamma \frac{1}{z^2+1}\,dz \quad \text{for } \gamma(t) = 2e^{it}, \quad 0 \le t \le 2\pi$$

(Hint: Rewrite the integrand using partial fractions.)

$**$ **54.**
Evaluate the integrals

$$\int_{\gamma_r} \frac{z^2+1}{z(z^2+4)}\,dz \quad \text{for } \gamma_r(t) = re^{it}, \quad 0 \le t \le 2\pi$$

for each r, with $0 < r < \infty$, but $r \ne 2$.

March 13

∗ **55.**
Let G be a domain in $\mathbb{C}$ and suppose f is a holomorphic function in G.

(a) Prove that if $f(z)$ is a real number for every z in G, then f is constant. (Hint: use the idea inherent in Problem 1. of the Test.)

(b) Prove that if L is any line in $\mathbb{C}$ and $f(z)$ is a point of L for every z in G then f is constant.

(c) Prove that if Γ is any circle in $\mathbb{C}$, that is, there are a in $\mathbb{C}$ and $R > 0$ so that $\Gamma = \{w : |w - a| = R\}$, and $f(z)$ is a point of Γ for every z in G then f is constant.

∗ **56.**
Use the result of problem **50.** and partial fractions to compute

$$\int_{-\infty}^{\infty} \frac{e^{ist}}{t^2 + 2t + 5}\, dt$$

for $s > 0$ and use your computation to evaluate the integrals

$$\int_{-\infty}^{\infty} \frac{\cos(st)}{t^2 + 2t + 5}\, dt \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin(st)}{t^2 + 2t + 5}\, dt$$

(Hint: Integrate over a path following (counterclockwise) the boundary of the region $\{z : 0 \leq |z| \leq R \text{ and } 0 \leq \arg(z) \leq \pi\}$ and then take the limit as R tends to ∞.)

∗ **57.**
Prove: If G is a domain in $\mathbb{C}$ and f is holomorphic and non-constant on G, then for each a in $\mathbb{C}$, the set $V_a = \{z \in G : f(z) = a\}$ is either $\emptyset$, a finite set, or a countably infinite set.

∗ **58.**
Prove: There is no function h that is holomorphic in the unit disk $\mathbb{D}$ and satisfies $f(1/n) = (-1)^n/n^2$ for $n = 2, 3, \cdots$.

59. The function φ is defined on the sequence $\{\frac{1}{n}\}$, for n a positive integer, by $\varphi(\frac{1}{n}) = \dfrac{2+n}{3+n^2}$
Is it possible to find an analytic function, ψ, defined on a neighborhood of 0 such that $\psi(\frac{1}{n}) = \varphi(\frac{1}{n})$ for all positive integers satisfying $n \geq N$ for some $N > 0$? If so, find the largest $R > 0$ such that ψ can be analytic on the disk $\{z : |z| < R\}$. If there are some functions satisfying this condition, how many are there? Explain your answers!

∗∗ **60.**
Prove: If g is an entire function such that for some positive integer n and there are positive numbers M and R so that $|f(z)| \leq M|z|^n$ for $|z| > R$, then f is a polynomial of degree less than or equal to n.

April 1

$*$ **61.**
Use Schwarz's Lemma and linear fractional 'changes of variables' to prove:
If f is a non-constant analytic function that maps the unit disk $\mathbb{D}$ into itself, then for each z and w in $\mathbb{D}$ with $z \neq w$, we have

$$\left|\frac{f(z)-f(w)}{1-\overline{f(w)}f(z)}\right| \leq \left|\frac{z-w}{1-\overline{w}z}\right|$$

Moreover, there is equality for some z_0 and w_0 in $\mathbb{D}$ if and only if f is a linear fractional map of the unit disk $\mathbb{D}$ onto itself.

$*$ **62.**
Let p be a polynomial of degree n and suppose $R > 0$ is large enough that $p(z) \neq 0$ for $|z| \geq R$. Prove that, if $\gamma(t) = Re^{it}$ for $0 \leq t \leq 2\pi$

$$\int_\gamma \frac{p'(z)}{p(z)}\,dz = 2\pi n i$$

$*$ **63.**
Evaluate the integral $\displaystyle\int_0^\infty \frac{1}{x^5+1}\,dx$ by integrating an analytic function along a curve that follows the boundary of the region $\{z : |z| \leq R \text{ and } 0 \leq \arg(z) \leq 2\pi/5\}$.

$*$ **64.**
The goal is to evaluate the integral $\displaystyle\int_0^\infty \frac{\sqrt{x}}{x^5+1}\,dx$ by imitating the strategy of Problem 63.
(a) Explain what theorem you used in #63 and how your use satisfies the hypotheses.
(b) Explain why you cannot use exactly the same contour in this problem.
(c) Use the contour associated with the region $\{z : \epsilon \leq |z| \leq R \text{ and } 0 \leq \arg(z) \leq 2\pi/5\}$, or a similar idea, to solve this problem.

$**$ **65.**
The smooth curves γ_0, γ_1, γ_2, and γ_3 all have $\gamma_j(0) = -i$ and $\gamma_j(1) = i$, with γ_0 following the imaginary axis, γ_1 following the shorter arc (between $-i$ and i) of the circle with center 1 and radius $\sqrt{2}$, γ_2 following the longer arc (between $-i$ and i) of the circle with center 1 and radius $\sqrt{2}$, and γ_3 following the longer arc ($-i$ to i) of the circle with center 2 and radius $\sqrt{5}$.

The function f is analytic in the disk centered at 0 with radius 10 and we know that the function f satisfies

$$\int_{\gamma_0} \frac{f(z)}{(z-2)(z-4)^2}\,dz = 7$$

Compute the integrals $\displaystyle\int_{\gamma_2} \frac{f(z)}{(z-2)(z-4)^2}\,dz$, $\displaystyle\int_{\gamma_2} \frac{f(z)}{(z-2)(z-4)^2}\,dz$, and $\displaystyle\int_{\gamma_3} \frac{f(z)}{(z-2)(z-4)^2}\,dz$ in terms of the given information and values of f, f', etc.

Along with giving the values of the integrals, give a careful explanation of the theorems you are using and especially the domains, the functions, and the curves involved in justifying your conclusions.

April 8

∗ **66.**
Let $f(z) = \dfrac{z^4 - 7z^2 - 5z - 8}{z^2 - z - 6}$

Find Laurent series for f that converge for $0 < |z| < 2$, for $2 < |z| < 3$, and for $3 < |z| < \infty$.

∗ **67.**
Prove: *Suppose f is holomorphic in a punctured disk with center z_0 and f has a pole of order m at z_0. Let $g(z) = (z - z_0)^m f(z)$ so that g has a removable singularity at z_0. Then $Res(f, z_0) = \frac{1}{(m-1)!} g^{(m-1)}(z_0)$.*

∗ **68.**
The function $g(z) = \dfrac{\cos(1/z)}{\sin(2/z)}$ has infinitely many singularities in $\mathbb{C}$.

Classify each of them as isolated/not-isolated and removable/pole of order –/essential. For each isolated singularity, find the residue of g at the singularity.

∗ **69.**
Evaluate:

$$\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4}\, dx$$

(Hint: Let γ be the boundary of $\{z : |z| \leq R$ and $0 \leq \arg z \leq \pi$.)

70. Evaluate:

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4 + x^2 + 1}\, dx$$

∗ **71.**
Evaluate:

$$\int_{0}^{\infty} \frac{\sin(x)}{x}\, dx$$

(Hint: Integrate $\dfrac{e^{iz}}{z}$ over γ, the boundary of $\{z : 0 < r \leq |z| \leq R$ and $0 \leq \arg z \leq \pi$.)

∗ **72.**
For $0 < a < 1$, evaluate:

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x}\, dx$$

April 24

$*$ **73.**
Change variables in Laplace's Equation from Cartesian coordinates to Polar coordinates: specifically, show that u defined in an open set in $\mathbb{R}^2$ is harmonic if and only if

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

$*$ **74.**
Suppose u is a real valued harmonic function in the open unit disk $\mathbb{D}$. Prove that u^2 is harmonic if and only if u is constant.

75. Suppose u is a complex valued harmonic function in the open unit disk $\mathbb{D}$. When is u^2 also harmonic?

$*$ **76.**
(a) Prove: *If u is the real part of the analytic function f, then* $\mathrm{Re}(f') = \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$.

(b) Show that $U(x, y) = \log(x^2 + y^2)$ is a harmonic function.

(c) Use the observation in (a) above to find the harmonic conjugate of U. What is the function $f = U + iV$?

$*$ **77.**
Use Rouche's Theorem to prove that, for $n \geq 1$ and a_0, a_1, $\cdots$, a_{n-1} arbitrary complex numbers, if

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$$

then $|P(z)| < 1$ for all z with $|z| = 1$ is impossible. (Hint: Write $P = f + g$.)

78. (a) Let H_0 be the half plane $H_0 = \{z : \mathrm{Re}(z) > 0\}$. For $f(z) = \sqrt{iz}$, where the branch of the square root has an appropriate domain and $\sqrt{1} = 1$, find $f(H_0)$.

(b) For $H_1 = \{z : \mathrm{Re}(z) > 1\}$, find $f(H_1)$.

(c) What curve forms the boundary of $f(H_1)$? Give a precise description!

$*$ **79.**
Let H_0 be the half plane $H_0 = \{z : \mathrm{Re}(z) > 0\}$, let $g(z) = \sqrt{z}$, where the branch of the square root has an appropriate domain and $\sqrt{1} = 1$, and let $h(z) = z^2$.
(a) Find a linear fractional map φ so that $\varphi(0) = -1$, $\varphi(1) = 0$, and $\varphi(\infty) = 1$.
(b) Describe the set $\varphi(g(h(z) + 1))$ for z in H_0.

$*$ **80.**
Let $\Omega = \{z : -\pi/2 < \mathrm{Re}(z) < \pi/2 \text{ and } \mathrm{Im}(z) > 0\}$.
(a) Find $\sin(\Omega)$.
(b) Find $\cos(\Omega)$.
(c) Find $\tan(\Omega)$.

For Discussion May 1

$*$ **81.**
Use the definition of infinite product to show that

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}$$

$*$ **82.**
Let $\mathbb{D}$ be the open unit disk, let $H_+ = \{z : \operatorname{Im}(z) > 0\}$, let $H_- = \{z : \operatorname{Im}(z) < 0\}$, and let $\Delta = \mathbb{D} \cap H_+$
(a) Find $f(\Delta)$ for $f(z) = z + z^{-1}$
(b) Find a function g defined, analytic, and univalent on $\mathbb{D}$ such that $g(\mathbb{D}) = \Delta$

$*$ **83.**
Let $\mathbb{D}$ be the open unit disk and let $\mathcal{S}$ be the interior of the square with vertices at $\pm 1 \pm i$, that is, $\mathcal{S} = \{w : -1 < \operatorname{Re}(w) < 1 \text{ and } -1 < \operatorname{Im}(w) < 1\}$.

Let $\sigma(z) = \sum_{n=0}^{\infty} a_n z^n$ be the (unique) analytic function on $\mathbb{D}$ such that σ is one-to-one on $\mathbb{D}$, $\sigma(\mathbb{D}) = \mathcal{S}$, $\sigma(0) = 0$, and $\sigma'(0) > 0$. Prove a_n is real for all n and that $a_n = 0$ unless $n = 4k + 1$ for some integer k.

$*$ **84.**
Suppose G is a bounded, connected, simply connected open subset of the plane such that there is a function ρ analytic and one-to-one on the disk $\{z : |z| < R\}$, with $R > 1$, for which $\rho(\mathbb{D}) = G$ and $\rho(\partial\mathbb{D})$ is a simple closed curve that forms the boundary of G.

Let a, b, c, and d be four distinct points on the boundary of G.
(a) Show that it is possible to find a function f holomorphic and univalent on $\mathbb{D}$ and continuous on $\mathbb{D} \cup \partial\mathbb{D}$ such that $f(\mathbb{D}) = G$ and $f(1) = a$ and $f(-1) = c$.
(b) When is it possible to find g holomorphic and univalent on $\mathbb{D}$ and continuous on $\mathbb{D} \cup \partial\mathbb{D}$ such that $g(\mathbb{D}) = G$ and $g(1) = a$, $g(i) = b$, and $g(-1) = c$?
(c) If it is possible to find g holomorphic and univalent on $\mathbb{D}$ and continuous on $\mathbb{D} \cup \partial\mathbb{D}$ such that $g(\mathbb{D}) = G$ and $g(1) = a$, $g(i) = b$, and $g(-1) = c$, when is also possible to find h, holomorphic and univalent on $\mathbb{D}$ and continuous on $\mathbb{D} \cup \partial\mathbb{D}$ such that $h(\mathbb{D}) = G$, and $h(1) = a$, $h(i) = b$, $h(-1) = c$, and $h(-i) = d$?

$*$ **85.**
Use a contour integral to evaluate $\displaystyle\int_0^{\infty} \frac{\log x}{x^2+9}\,dx$. Give careful definitions of your contours. If you use a complex logarithm function, define the branch you are using.

$*$ **86.**
Suppose q is analytic on the disk $\{z : |z| < R\}$ where $R > 1$.

(a) What is the order of the zero of $q - q(0) - q'(0)z$ at the origin?

(b) Prove: If $|q(0)| + |q'(0)| < \min\{|q(z)| : |z| = 1\}$, then q has at least two zeros (counting multiplicity) in $\mathbb{D}$.