The Implicit Function Theorem (c.f. 2.10.10 and 2.10.13)

Let U be an open set in $\mathbb{R}^{n+m}$ and let $F : U \mapsto \mathbb{R}^n$ be a continuously differentiable function from U into $\mathbb{R}^n$. If c is a point of U such that $F(c) = 0$ and such that $(DF)(c)$ maps $\mathbb{R}^{n+m}$ onto $\mathbb{R}^n$, that is, such that $(DF)(c)$ has rank n, then in a neighborhood of c, $F(x) = 0$ determines n of the $n+m$ variables in terms of the other m variables. In particular, if the columns of $(DF)(c)$ corresponding to $x_1, x_2, \cdots, x_n$ are linearly independent, then in some neighborhood of $\hat{c} = (c_{n+1}, c_{n+2}, \cdots, c_{m+n})$, there are continuously differentiable functions $\varphi_1, \varphi_2, \cdots, \varphi_n$ of $x_{n+1}, x_{n+2}, \cdots, x_{n+m}$ such that

$$F(\varphi_1, \varphi_2, \cdots, \varphi_n, x_{n+1}, x_{n+2}, \cdots, x_{m+n}) = 0$$

and for which

$$\frac{\partial(\varphi_1, \varphi_2, \cdots, \varphi_n)}{\partial(x_{n+1}, x_{n+2}, \cdots, x_{m+n})} = -L_n^{-1} L_m$$

where $(DF)(x) = L = (L_n\ L_m)$, that is, L_n is the $n \times n$ matrix consisting of the first n columns of $L = (DF)(x)$ and L_m is the $n \times m$ matrix consisting of the last m columns of L.

The classic example from Freshman calculus is that of the unit circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$ determining y as a function of x in the upper and lower semi-circles, or x as a function of y in the right and left semicircles. We can write $F(x, y) = x^2 + y^2 - 1$, so that the circle becomes the set $F(x, y) = 0$. This function has $(DF)(x, y) = (2x\ 2y)$ and this Jacobian matrix has rank 1 at each point of $\mathbb{R}^2$ except $(0, 0)$ which is not on the circle. Then, near the point $c = (.6, -.8)$ which satisfies $F(c) = 0$, $(DF)(c) = (1.2, -1.6)$ has rank 1, and the first 1 column is linearly independent, so we can write x as a continuously differentiable function of y in a neighborhood of $\hat{c} = -.8$. In this case, $x(y) = \varphi_1(y) = \sqrt{1 - y^2}$. As the Theorem says, $F(\varphi_1(y), y) = 0$ and

$$\frac{d\varphi_1}{dy} = -\left(\frac{\partial F}{\partial x}\right)^{-1}\left(\frac{\partial F}{\partial y}\right) = -(2x)^{-1}(2y) = -\frac{y}{x} = -\frac{y}{\sqrt{1-y^2}}$$

Notice that the second column of $(DF)(c)$ is also linearly independent, so we can also find y as a function of x for x near .6. Rewriting F as $G(y, x) = x^2 + y^2 - 1$, where we are putting y in the first position to agree with the Hubbards' notation in the statement of the Theorem, we have $(DG)(y, x) = (2y\ 2x)$ and $y(x) = \psi_1(x) = -\sqrt{1 - x^2}$ is continuously differentiable in a neighborhood of $x = .6$. As the Theorem says, $G(\psi_1(x), x) = 0$ and

$$\frac{d\psi_1}{dx} = -\left(\frac{\partial G}{\partial y}\right)^{-1}\left(\frac{\partial G}{\partial x}\right) = -(2y)^{-1}(2x) = -\frac{x}{y} = \frac{x}{\sqrt{1-x^2}}$$

Finally, notice that at the four points $(\pm 1, 0)$ and $(0, \pm 1)$ on the circle, because the Jacobian $(DF)(x, y)$ is $(\pm 2\ 0)$ or $(0\ \pm 2)$ which are rank 1, but with only one of the columns linearly independent, the Theorem does not guarantee that each variable can be written, nearby, as a function of the other, and, indeed, they cannot.

1. Show there are functions $w = \varphi_1(y, z)$ and $x = \varphi_2(y, z)$ that satisfy $F(\varphi_1, \varphi_2, y, z) = 0$ and find their derivatives for $F(w, x, y, z) = \begin{pmatrix} 3w + 2x - y + 3z - 2 \\ 2w + x + 5y - 2z - 3 \end{pmatrix}$ near $c = (0, 0, 1, 1)$.

2. Show there is a function $z = \varphi_1(x, y)$ satisfying $F(\varphi_1, x, y) = 0$ and find its derivative for the hyperboloid of two sheets determined by $F(z, x, y) = x^2 - 6x + y^2 + 4y - z^2 + 8z$ near $c = (0, 0, 0)$.

3. Show there are functions $x = \varphi_1(z)$ and $y = \varphi_2(z)$ satisfying $F(\varphi_1, \varphi_2, z) = (0, 0)$ and find their derivatives for the 'tennis ball curve' determined by $F(x, y, z) = \begin{pmatrix} x^2 + y^2 + z^2 - 14 \\ x^2 - y^2 - z \end{pmatrix}$ near $c = (2, -1, 3)$.