Math 44500 (Cowen) **Handout 3** **20 January 2011**

Due Thursday, 27 January:

From page 118 of **BS**: 4, 9 and prove $\lim_{x\to\infty} \frac{3x-7}{x+5} = 3$

A. If the function g maps X into itself and a is a point of X, we say *a is a fixed point of g* if $g(a) = a$. Suppose f is a continuous function on $[0,1]$ and $f([0,1]) \subset [0,1]$. Use the strategy given in (a)-(d) to show that f has a fixed point in $[0,1]$. (This is a special case of the *Brouwer Fixed Point Theorem* for $\mathbb{R}^n$.)

(a) Let $L = \{x \in [0,1] : f(x) < x\}$ and let $G = \{x \in [0,1] : f(x) > x\}$. Show that each of L and G are open subsets of $[0,1]$, that is, there are L' and G' open subsets of $\mathbb{R}$ such that $L = L' \cap [0,1]$ and $G = G' \cap [0,1]$.

(b) Show that $L \cap G = \emptyset$.

(c) Show that $L \cup G \neq [0,1]$.

(d) Finish proof.

(e) Notice that each of the following are continuous functions that map $[0,1]$ into itself. Find fixed points (or approximations to three decimal places) of each of the functions:

(i) $f(x) = 1 - x$ (ii) $g(x) = 1 - 3x + 3x^2$
(iii) $h(x) = 1/(3-x)$ (iv) $c(x) = \cos(x)$

B. Compare with **A:** Suppose f is a continuous function on $[0,\infty)$ with $f([0,\infty)) \subset [0,\infty)$. Where does the proof you gave of **A.** break down? Give an example of a continuous function mapping $[0,\infty)$ into itself with no fixed point.

Definition: Suppose f is a continuous function on $[0,1]$ and $f([0,1]) \subset [0,1]$. We define the *iterates of f* by $f_{(0)}(x) = x$, $f_{(1)}(x) = f(x)$, $f_{(2)}(x) = f(f(x))$, $f_{(3)}(x) = f(f(f(x)))$, and more generally, for j in $\mathbb{N}$, let $f_{(j+1)}(x) = f(f_{(j)}(x))$.

Convince yourself of the following facts:

- For each j in $\mathbb{N}$, the function $f_{(j)}$ is a continuous function mapping $[0,1]$ into itself.
- The collection $\{f_{(j)} : j \in \mathbb{N}\}$ is a (discrete) semi-group: $f_{(j)} \circ f_{(k)} = f_{(j+k)}$.

C. Compare with **A:**

(a) A contractive mapping is a mapping that satisfies, for some α with $0 < \alpha < 1$, that $|f(x) - f(y)| \leq \alpha|x-y|$. If f is a map of $[0,1]$ into itself that is contractive, then $a = \lim_{j\to\infty} f_{(j)}(0)$ exists and is the unique fixed point of f in $[0,1]$.

(b) If g is a continuous, increasing function of $[0,1]$ into $[0,1]$, then then $a = \lim_{j\to\infty} g_{(j)}(0)$ exists and is a fixed point of g in $[0,1]$.

(c) Find two fixed points of the (increasing) function $g(x) = x^2$.

D. Explain why the following strategy of built on the idea of **C.** does *NOT* work for proving the fixed point theorem of **A.**: Suppose f is a continuous function mapping $[0,1]$ into itself. For any point p of $[0,1]$ (for example $p = 0$), the sequence $f_{(j)}(p)$ has a convergent subsequence in $[0,1]$ and this subsequence should converge to a fixed point of f.