1. Let $\mathbb{Z}$ denote the set of integers. Prove that $d(m,n) = |m-n|$ defines a metric on $\mathbb{Z}$.

2. Suppose (X,d) is a metric space. Let D be the non-negative function on $X \times X$ be defined by

$$D(x,y) = \begin{cases} d(x,y) & \text{for } d(x,y) < 1 \\ 1 & \text{for } d(x,y) \geq 1 \end{cases}$$

(a) Prove that D is also a metric on X, so that (X,D) is also a metric space.

(b) Prove that the sequence (x_j) in X converges to w in the metric d on X if and only if the sequence (x_j) in X converges to w in the metric D.
(This means that the metric spaces (X,d) and (X,D) have the same convergent sequences.)

(c) Show that the metric space (X,d) is a complete metric space if and only if (X,D) is a complete metric space.

3.(a) Show that the function $f(t) = \dfrac{t}{1+t}$ is an increasing function on $[0,\infty)$.

(b) Prove that $\Delta(x,y) = \dfrac{|x-y|}{1+|x-y|}$ is a metric on $\mathbb{R}$.

4. Suppose (X,d) is a metric space. For c in X and $r > 0$, let $B(c,r)$ be defined by
$B(c,r) = \{x \in X : d(x,c) < r\}$, the '*open*' ball with center c and radius r, and
let $F(c,r) = \{x \in X : d(x,c) \leq r\}$, be the '*closed*' ball with center c and radius r.

(a) Show that, for every c in X and $r > 0$, the set $B(c,r)$ is an open set in X.

(b) Show that, for every c in X and $r > 0$, the set $F(c,r)$ is a closed set in X.

5. Suppose X is a set and $\mathcal{T}$ is a topology on X. Recall that this means that $(X,\mathcal{T})$ is a topological space and that the open sets in this space are the subsets of X that are in $\mathcal{T}$, and the closed sets in this space are the sets $F = X \setminus U = U^c$ where U is an open set.

(a) Show that if F_1, F_2, $\cdots$, F_k are closed sets in X, then $\displaystyle\bigcup_{j=1}^{k} F_j$ is also a closed set in X.

(b) Show that if J is an index set and F_j is a closed set in X for each j in J, then $\displaystyle\bigcap_{j\in J} F_j$ is also a closed set in X.

Definition: Supppose X is a topological space. For a subset S of X, *the closure of* S in X, denoted $\overline{S}$, is the intersection of all the closed sets in X that contain S.

By part (b) of Exercise 4, the closure $\overline{S}$ is a closed set for any set S and, indeed, $\overline{S}$ is the smallest closed set containing S.

6.(a) Consider $\mathbb{R}^2$ as a metric space using the usual distance as the metric. For c a point in $\mathbb{R}^2$ and $r > 0$, prove that the closed ball $F(c,r)$ is the closure of the open ball $B(c,r)$, that is, prove that $\overline{B(c,r)} = F(c,r)$.

(b) Decide whether the conclusion in part (a) is true for all metric spaces. That is, either prove that if (X,d) is any metric space, c is any point in X and $r > 0$, then $\overline{B(c,r)} = F(c,r)$,
OR find an example of a metric space X, a point c in X, and a number $r > 0$ so that $\overline{B(c,r)} \neq F(c,r)$.