Homework 11

1. The vectors $v_1 = (1, 1, -1)$; $v_2 = (2, 1, 2)$; and $v_3 = (2, -1, -1)$ are a basis for $\mathbb{R}^3$. Use the Gram–Schmidt orthogonalization process to create an orthonormal basis for $\mathbb{R}^3$.

2. The vectors $v_1 = (1, 1, -1, 1)$; $v_2 = (1, 0, 1, 2)$; $v_3 = (1, -2, -1, 0)$; and $v_4 = (0, 2, 1, -1)$ are a basis for $\mathbb{R}^4$. Use the Gram–Schmidt orthogonalization process to create an orthonormal basis for $\mathbb{R}^4$.

3. The vectors $v_1 = (1, 0, -1)$ and $v_2 = (2, 1, -1)$ span the subspace $\mathcal{U}$ in $\mathbb{R}^3$. Use the Gram–Schmidt orthogonalization process to create an orthonormal basis for $\mathcal{U}$.

4. The vectors $v_1 = (1, 0, -1, 1)$; $v_2 = (2, 1, 1, -1)$; and $v_3 = (1, -1, -1, 0)$ span the subspace $\mathcal{W}$ in $\mathbb{R}^4$. Use the Gram–Schmidt orthogonalization process to create an orthonormal basis for $\mathcal{W}$.

5. Let $\mathcal{W}$ be the hyperplane (i.e. 3 dimensional subspace) in $\mathbb{R}^4$ with equation $2a + b - c + 2d = 0$. Find an orthonormal basis for $\mathcal{W}$.

6. Let M be the subspace of $\mathbb{R}^4$ spanned by $v_1 = (1, 1, 0, 0)$; $v_2 = (0, 1, 1, 0)$; $v_3 = (0, 0, 1, 1)$; and $v_4 = (1, 0, 0, 1)$.
   (a) Use the Gram-Schmidt orthogonalization process to find an orthonormal basis for M.
   (b) Extend the basis you found in part (a) to an orthonormal basis for all of $\mathbb{R}^4$.
   (c) Is $w = (1, -1, 0, 1)$ in M? (Justify your answer.)

7. Let $C = \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 2 & -1 & 5 & -4 \end{pmatrix}$ (Compare with problem 2. of Homework 8).
   (a) Find an orthonormal basis for the nullspace of C.
   (b) Find an orthonormal basis for the range of C'.
   (c) Find the angles between the vectors you found in parts (a) and (b).
   (d) Find an orthonormal basis for the nullspace of C'.
   (e) Find an orthonormal basis for the range of C.
   (f) What do you notice about your answers to parts (d) and (e).

8. Let M be the subspace of $\mathbb{R}^4$ spanned by $(2, 1, 0, -1)$, $(1, 1, 1, 0)$, and $(-1, 0, 1, 1)$.
   Find w in M and u in $M^\perp$ so that $w + u = (3, 1, 0, 0)$.

9. Let $\mathcal{W}$ be the subspace of $\mathbb{R}^5$ spanned by $(1, 1, 0, -1, 1)$, $(0, 2, -1, 1, 1)$, and $(-1, 1, 1, -2, 1)$.
   Find w in $\mathcal{W}$ and u in $\mathcal{W}^\perp$ so that $w + u = (1, 1, -1, 1, 1)$.

10. The matrix B is an 8×11 matrix and the dimension of $\mathcal{N}(B)$, the nullspace of B, is 5.
    (a) What is the dimension of $\mathcal{R}(B)$, the range of B?
    (b) What is the dimension of $\mathcal{R}(B')$, the range of B'?
    (c) What is the dimension of $\mathcal{N}(B')$, the nullspace of B'?
    (d) What is the dimension of $\mathcal{R}(B')^\perp$, the orthogonal complement of $\mathcal{R}(B')$?

11. An $n \times n$ matrix U is called *unitary* if $U' = U^{-1}$.
    (a) Show U is unitary if and only if its columns form an orthonormal basis for $\mathbb{C}^n$.
    (b) Show that if U is unitary, then U^{-1} is also unitary.
    (c) Show that if U is a real unitary matrix, the map $x \mapsto Ux$ is a rigid motion of $\mathbb{R}^n$ by showing that if U is unitary and v and w are vectors in $\mathbb{R}^n$, then
        (i) $\langle Uv, Uw\rangle = \langle v, w\rangle$,
        (ii) $\|Uv\| = \|v\|$,
        (iii) the angle between Uv and Uw is the same as the angle between v and w.