Examples

1. Find vectors that span $\mathcal{N}(C)$ the nullspace of the matrix $C = \begin{pmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & 0 \\ 3 & 0 & 1 & 2 \end{pmatrix}$

2. Find vectors that span $\mathcal{R}(C)$, the range of the matrix $C = \begin{pmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & 0 \\ 3 & 0 & 1 & 2 \end{pmatrix}$

3. Let $B = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ -1 & 0 & 1 \end{pmatrix}$.
   (a) Show that $(4, -3, -3)$ is a linear combination of $(1, 2, -1)$, $(1, -1, 0)$, and $(-1, 2, 1)$.
   (b) The vector $v = (1, 2, -1) + 2(1, -1, 0) - (-1, 2, 1) = (4, -2, -2)$ is a linear combination of the columns of B. Find X so that $BX = v$.
   (c) Is the vector $(-2, 11, 1)$ in the subspace spanned by $(1, 2, -1)$, $(1, -1, 0)$, and $(-1, 2, 1)$?

4. In answering a question on her linear algebra homework, April claimed that the subspace $\mathcal{W}$ is spanned by the set $u_1 = (1, 0, 1)$ and $u_2 = (0, 1, -1)$. Michelle claimed that the subspace $\mathcal{W}$ is spanned by $v_1 = (1, 1, 0)$, $v_2 = (2, 1, 1)$, and $v_3 = (1, -1, 2)$. Do their answers agree with each other, that is, is the subspace spanned by the set $\{u_1, u_2\}$ the same as the subspace spanned by $\{v_1, v_2, v_3\}$?

5. In answering a question on his linear algebra homework, Max claimed that the subspace $\mathcal{U}$ is spanned by the set $u_1 = (1, 0, 1, 1)$, $u_2 = (0, 1, -1, 0)$ and $u_3 = (0, 0, 1, 2)$. Spike claimed that the subspace $\mathcal{U}$ is spanned by $v_1 = (1, 1, 0, 1)$, $v_2 = (2, 1, 1, 2)$, and $v_3 = (1, -1, 2, 1)$. Do they agree with each other, that is, is the subspace spanned by the set $\{u_1, u_2, u_3\}$ the same as the subspace spanned by $\{v_1, v_2, v_3\}$?

   Decide if the following sets of vectors are linearly dependent or independent. If independent, prove that they are, if dependent, find a non-trivial linear combination of the vectors that gives zero.

6. $\{(0, 1, 1, -1),\ (1, 3, 1, -2),\ (2, 1, 0, -3),\ (3, 1, -1, 2),\ (2, -1, 2, 0)\}$

7. $\{(1, 1, -1, 2),\ (3, -1, 1, 1),\ (2, 0, -1, 1),\ (0, 2, -3, 2)\}$

8. The vectors $v_1 = (1, -1, 2)$, $v_2 = (-1, 2, -3)$, $v_3 = (1, 1, -1)$, and $v_4 = (-2, 3, -4)$ are linearly dependent in $\mathbb{R}^3$. Write one of the vectors as a linear combination of the rest.

9. Find a basis for the solution space of the system:
$$\begin{cases} u + 3v - w + 2x + y = 0 \\ u + 2v + 4w + 2x = 0 \\ 2u + 8v + w + 3x - y = 0 \end{cases}$$
   What is the dimension of this subspace?

10. $3x + 2y - z = 0$ is the equation of a plane $\mathcal{P}$ in $\mathbb{R}^3$ that passes through the origin, so $\mathcal{P}$ is a subspace of $\mathbb{R}^3$. Find a basis for $\mathcal{P}$. What is the dimension of this subspace?

11. Let $A = \begin{pmatrix} 1 & -1 & 1 & 0 & -2 \\ 2 & 0 & 1 & 1 & -1 \\ 1 & -3 & 2 & -1 & -5 \end{pmatrix}$
    (a) Find a basis for the column space of A.
    (b) Find a basis for the null space of A.
    (c) Is $(1,1,1)$ in the column space of A?

12. Let $B = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -3 \\ 3 & 1 & 5 \\ 2 & -1 & 5 \end{pmatrix}$
    (a) Find a basis for the column space of B.
    (b) Find a basis for the column space of B'.
    (c) Is $(1,3,-1,-4)$ in the column space of B?

13. Find a basis for the subspace of $\mathbb{R}^4$ spanned by $(1,-1,5,-5)$, $(1,1,-1,1)$, and $(2,1,1,-1)$. What is the dimension of this subspace?

14. Find a basis for the subspace of $\mathbb{R}^4$ spanned by $(1,-1,1,2)$, $(1,1,-1,1)$, and $(2,1,1,-1)$. What is the dimension of this subspace?

15.
$$C = \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 2 & -1 & 5 & -4 \end{pmatrix}$$
    (a) What is the dimension of $\mathcal{N}(C)$, the null space of C? What is the rank of C? What is the rank of C'? What is the dimension of $\mathcal{N}(C')$?
    (b) Find bases for the range of C, the range of C', and the nullspace of C'.
    (c) Is the equation $CX = (2,3,1)$ solvable?
    (d) Is the equation $CX = (-1,1,3)$ solvable?
    (e) Is the equation $C'X = (1,-1,2,-3)$ solvable?

16. Find bases for the range and the nullspace, and find the dimensions of these subspaces.
$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & -1 & 4 & 0 \end{pmatrix}$$

17. Find the general solution of the system, and find two explicit solutions, if the system has infinitely many solutions:
$$\begin{cases} u + 2v + w - x - 2y &= 3 \\ -2u + v + w + x + 2y &= 5 \\ u + v - w + 2x + 4y &= -2 \\ u - v \quad\quad + 3x + y &= -7 \\ -u + 3v + w + x + 3y &= 7 \end{cases}$$

18. Find the general solution of the system, and find two explicit solutions, if the system has infinitely many solutions:
$$\begin{cases} a + b + 4c + d + e &= 8 \\ a - b + 2c + 2d + e &= 1 \\ 2a + b - c - d - 2e &= 4 \\ b + 3c + d + e &= 5 \\ 2a - b + c + 3d &= 0 \end{cases}$$