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LINEAR FRACTIONAL COMPOSITION OPERATORS ON H^2

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Abstract. If φ is an analytic function mapping the unit disk D into itself, the composition operator C_φ is the operator on H^2 given by $C_\varphi f = f \circ \varphi$. The structure of the composition operator C_φ is usually complex, even if the function φ is fairly simple. In this paper, we consider composition operators whose symbol φ is a linear fractional transformation mapping the disk into itself. That is, we will assume throughout that

$$\varphi(z) = \frac{az+b}{cz+d}$$

for some complex numbers a, b, c, d such that φ maps the unit disk D into itself. For this restricted class of examples, we address some of the basic questions of interest to operator theorists, including the computation of the adjoint.

For any φ that maps the disk into itself, it is known that C_φ is a bounded operator, and some general properties of C_φ have been established (see for example, [15], [12], [17], [13], [10], [3], [11], [14], and [16]). However, not all questions that would be considered basic by operator theorists are understood. For example, for general φ, no convenient description of C_φ^* is known and it is not known how to compute $\|C_\varphi\|$ (although order of magnitude estimates are available [3]).

J. S. Shapiro (see [16]) has completely answered the question "When is C_φ compact?" Although the general answer is complicated, if φ is a linear fractional transformation C_φ is compact if and only if φ maps the closed unit disk into the open disk. It follows from this that for a linear fractional φ, all powers of C_φ are non-compact if and only if φ has a fixed point on the unit circle.

The first section illustrates the diversity of this class of examples by showing there are eight distinct classes on the basis of spectral information alone. Much of the spectral information depends on the behavior of φ near the *Denjoy-Wolff* point, the unique fixed point $\hat{\alpha}$ of φ in the closed disk such that $|\varphi'(\hat{\alpha})| \le 1$.

In the second section of the paper, we find that in the linear fractional case C_φ^* is the product of Toeplitz operators and another composition operator. From this computation, we derive $\|C_\varphi\|$ in certain cases and give a short proof of the subnormality of C_φ^* when φ is a hyperbolic inner linear fractional transformation (see also [14, 5]). Finally, the class of linear fractional transformations for which C_φ is hyponormal or subnormal is identified.

The class of composition operators is related to other areas of operator theory in somewhat surprising ways. For example, Deddens [6] established a connection between the discrete Cesaro operator and C_φ where $\varphi(z) = sz + 1 - s$ for $0 < s < 1$ and showed that therefore C_φ^* is subnormal for these φ. In addition, commutants of many analytic Toeplitz operators are generated by composition and multiplication operators.

Although this paper makes progress in answering some basic questions about linear fractional composition operators, there are still problems to be considered. For example, computing the norm is still unsolved except in special cases and exact conditions for unitary equivalence and similarity are not known. It is hoped that the results here will point the way toward results about more general composition operators, both on H^2 and on related Hilbert spaces of analytic functions.

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Eight Examples

In spite of their apparent simplicity, composition operators on H^2 with linear fractional symbol exhibit great diversity. In the following table, we collect some examples that show that linear fractional transformations give rise to most of the major spectral types. The Denjoy-Wolff point will be denoted $\hat{\alpha}$.

Example	*Properties*	*Spectrum* C_φ	*Reference*
$\varphi(z) = \zeta z$ where $\lvert\zeta\rvert = 1$	$\hat{\alpha} = 0$ elliptic, inner	closure$\{\zeta^k : k = 0, 1, \dots\}$	[12]
$\varphi(z) = \frac{3z+1}{z+3}$	$\hat{\alpha} = 1,\ \varphi'(1) = \frac{1}{2}$ hyperbolic, inner	$\{\lambda : \frac{1}{\sqrt{2}} \le \lvert\lambda\rvert \le \sqrt{2}\}$	[12]
$\varphi(z) = \frac{(1+2i)z-1}{z-1+2i}$	$\hat{\alpha} = 1,\ \varphi'(1) = 1$ parabolic, inner	$\{\lambda : \lvert\lambda\rvert = 1\}$	[12]
$\varphi(z) = sz + 1 - s$ where $0 < s < 1$	$\hat{\alpha} = 1,\ \varphi'(1) = s$	$\{\lambda : \lvert\lambda\rvert \le \frac{1}{\sqrt{s}}\}$	[6] [3, Cor. 4.8]
$\varphi(z) = \frac{(2-t)z+t}{-tz+2+t}$ where $\mathrm{Re}(t) > 0$	$\hat{\alpha} = 1,\ \varphi'(1) = 1$ $\varphi''(1) = t$	$\{e^{\beta t} : \beta \le 0\} \cup \{0\}$ (a spiral)	[3, Cor. 6.2]
$\varphi(z) = \frac{rz}{1-(1-r)z}$ where $0 < r < 1$	$\hat{\alpha} = 0,\ \varphi'(0) = r$ $\varphi(1) = 1$	$\{\lambda : \lvert\lambda\rvert \le \sqrt{r}\} \cup \{1\}$	[10, Th. 3.8]
$\varphi(z) = -\frac{1}{2}z + \frac{1}{2}$	$\hat{\alpha} = \frac{1}{3},\ \varphi'(\frac{1}{3}) = -\frac{1}{2}$ C_φ^2 compact, C_φ not	$\{(-\frac{1}{2})^k : k = 0, 1, \dots\} \cup \{0\}$	[1]
$\varphi(z) = -\frac{1}{2}z$	$\hat{\alpha} = 0,\ \varphi'(0) = -\frac{1}{2}$ C_φ compact	$\{(-\frac{1}{2})^k : k = 0, 1, \dots\} \cup \{0\}$	[1]

Adjoints

If φ is an inner function in H^∞ so that it may be considered to be a mapping of the unit circle into itself, then a satisfactory formula can easily be obtained for C_φ^* by changing variables in the integral giving the inner product. The formula is more or less simple depending on the multiplicity of φ, but in any case, the operator can be described as a weighted expectation operator. If φ is a general analytic function mapping D into itself, no satisfactory formula for C_φ^* is known. In this section, we obtain a simple formula for C_φ^* when φ is a linear fractional transformation. The adjoint is a product of Toeplitz operators and a composition operator.

LEMMA 1. *If $\varphi(z) = (az + b)(cz + d)^{-1}$ is a linear fractional transformation mapping D into itself, where $ad - bc = 1$, then $\sigma(z) = (\overline{a}z - \overline{c})(-\overline{b}z + \overline{d})^{-1}$ maps D into itself.*

PROOF. Linear fractional transformations may be regarded as one-to-one mappings of the Riemann sphere onto itself. Let $\check{D}$ denote the open set

$$\check{D} = \{z : |z| > 1\} \cup \{\infty\}.$$

Now, φ maps D into itself, so $\gamma(z) = \overline{\varphi(\overline{z})}$ also maps D into itself. It follows that $\gamma^{-1}(z)$ maps $\check{D}$ into itself. An easy calculation shows that

$$\sigma(z) = \frac{1}{\gamma^{-1}(\frac{1}{z})}$$

which implies $\sigma(z)$ maps D into D. ■

Recall that for g in $L^\infty(\partial D)$, the Toeplitz operator T_g is the operator on H^2 given by $T_g(f) = Pfg$ for f in H^2 and P the orthogonal projection of L^2 onto H^2. (For general properties of Toeplitz operators, see [7, Chapter 7].)

THEOREM 2. *Let $\varphi(z) = (az+b)(cz+d)^{-1}$ be a linear fractional transformation mapping D into itself, where $ad - bc = 1$.*
Then $\sigma(z) = (\overline{a}z - \overline{c})(-\overline{b}z + \overline{d})^{-1}$ maps D into itself, $g(z) = (-\overline{b}z + \overline{d})^{-1}$ and $h(z) = cz + d$ are in H^∞, and

$$C_\varphi^* = T_g C_\sigma T_h^*.$$

PROOF. The function h is clearly in H^∞. By Lemma 1, σ maps D into itself and since the denominators of σ and g are the same, g is in H^∞. This means the formula makes sense.

Now, for α in D, let $K_\alpha(z) = (1 - \overline{\alpha}z)^{-1}$. This function is the reproducing kernel at α, that is, $\langle f, K_\alpha \rangle = f(\alpha)$ for f in H^2. It is easily proved that $T_h^* K_\alpha = \overline{h(\alpha)} K_\alpha$ and $C_\varphi^* K_\alpha = K_{\varphi(\alpha)}$. Calculation gives

$$\begin{aligned} T_g C_\sigma T_h^*(K_\alpha)(z) &= \overline{h(\alpha)} T_g C_\sigma (K_\alpha)(z) \\ &= \overline{(c\alpha + d)} \left(\frac{1}{-\overline{b}z + \overline{d}} \right) \left(\frac{1}{1 - \overline{\alpha} \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}} \right) \\ &= \frac{\overline{c\alpha} + \overline{d}}{-\overline{b}z + \overline{d} - \overline{\alpha a}z + \overline{\alpha c}} \\ &= \frac{1}{1 - \overline{\varphi(\alpha)}z} = K_{\varphi(\alpha)}(z) = C_\varphi^*(K_\alpha)(z). \end{aligned}$$

Since the K_α span a dense set of H^2, the desired equality holds. ■

A Norm Calculation

The best general estimate of the norm of C_φ is

$$\frac{1}{\sqrt{1 - |\varphi(0)|^2}} \le \|C_\varphi\| \le \frac{1 + |\varphi(0)|}{\sqrt{1 - |\varphi(0)|^2}}$$

and both inequalities can be achieved by linear fractional transformations [3, page 81].

In this section, we use the adjoint calculation of the previous section to find the norm of the composition operators with affine symbol. It will become clear that the norms of composition operators depend in a rather complex way on the parameters of the symbol.

THEOREM 3. *If $\varphi(z) = sz + t$ for $|s| + |t| \le 1$, then*

$$\|C_\varphi\| = \sqrt{\frac{2}{1 + |s|^2 - |t|^2 + \sqrt{(1 - |s|^2 + |t|^2)^2 - 4|t|^2}}}.$$

PROOF. In the trivial cases $s = 0$ and $t = 0$, the formula gives the correct norm $\|C_\varphi\| = (1 - |t|^2)^{-\frac{1}{2}}$, [3, page 81]. We therefore assume s and t are non-zero.

The function φ has not been presented in a way that we may directly apply the adjoint calculation; choosing $a^2 = s$ and $b = t/a$, a normalized expression for φ is $\varphi(z) = (az + b)(0z + a^{-1})^{-1}$. In the notation of theorem 2, $C_\varphi^* = T_g C_\sigma T_h^*$ where

$$g(z) = \left(-\bar{b}z + \bar{a}^{-1}\right)^{-1},$$

$$\sigma(z) = \frac{\bar{a}z}{-\bar{b}z + \bar{a}^{-1}},$$

$$\text{and } h(z) = a^{-1}.$$

Thus,

$$C_\varphi^* C_\varphi = \bar{a}^{-1} T_g C_\sigma C_\varphi = T_{(1-\overline{ab}z)^{-1}} C_\sigma C_\varphi = T_f C_\psi$$

where $f(z) = (1 - \bar{t}z)^{-1}$ and

$$\psi(z) = \varphi(\sigma(z)) = \frac{(|s|^2 - |t|^2)z + t}{-\bar{t}z + 1}.$$

Now

$$\begin{aligned} \|C_\varphi\|^2 &= \|C_\varphi^* C_\varphi\| = \lim_{n\to\infty} \|(C_\varphi^* C_\varphi)^n\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \|T_f T_{f\circ\psi} \cdots T_{f\circ\psi_{n-1}} C_\psi^n\|^{\frac{1}{n}} \\ &\leq \lim_{n\to\infty} (\|f\|_\infty \|f\circ\psi\|_\infty \cdots \|f\circ\psi_{n-1}\|_\infty)^{\frac{1}{n}} \lim_{n\to\infty} \|C_\psi^n\|^{\frac{1}{n}}. \end{aligned}$$

(Here ψ_k denotes the k^{th} iterate of ψ.) The last quantity in this expression is just the spectral radius of C_ψ which was calculated in [3, Theorem 2.1]. If $|s| + |t| = 1$, then $\psi(t/|t|) = t/|t|$ and $\psi'(t/|t|) = 1$, so the spectral radius of C_ψ is 1. If $|s| + |t| < 1$, then ψ maps the closed disk into the open disk and C_ψ is compact and has spectral radius 1. Thus, the last quantity in this expression is always 1.

In the case $|s| + |t| = 1$, since $0 < |t| < 1$, we find

$$|\psi(-t/|t|)| = \left|3 - \frac{4}{1 + |t|}\right| < 1.$$

This information, together with the fact from the above paragraph that the Denjoy-Wolff point of ψ is $t/|t|$, implies that ψ maps the closed unit disk onto a proper subdisk internally tangent to the unit circle at $t/|t|$. In particular, this means that $\lim_{n\to\infty} \psi_n(z) = t/|t|$, uniformly, in the closed unit disk (see, for example, [2]). It follows that $\lim_{n\to\infty} f\circ\psi_n = (1 - |t|)^{-1} = |s|^{-1}$ so

$$\lim_{n\to\infty} (\|f\|_\infty \|f\circ\psi\|_\infty \cdots \|f\circ\psi_n\|_\infty)^{\frac{1}{n}} = |s|^{-1}.$$

The above inequality now implies that $\|C_\varphi\| \leq |s|^{-\frac{1}{2}}$.

On the other hand, taking $\alpha = rt|s|/(s|t|)$, we find

$$\begin{aligned} \|C_\varphi\|^2 &\geq \lim_{r\to1^-} \frac{\|C_\varphi^* K_\alpha\|^2}{\|K_\alpha\|^2} = \lim_{r\to1^-} \frac{\|K_{\varphi(\alpha)}\|^2}{\|K_\alpha\|^2} \\ &= \lim_{r\to1^-} \frac{1 - r^2}{1 - (r|s| + |t|)^2} = \frac{1}{|s|}. \end{aligned}$$

Thus, $\|C_\varphi\| = |s|^{-\frac{1}{2}}$ when $|s| + |t| = 1$, which agrees with the conclusion for this case.

In the case $|s| + |t| < 1$, then $\psi(z) = (pz + t)(-\bar{t}z + 1)^{-1}$, where $p = |s|^2 - |t|^2$, and the fixed point of ψ in D is the smaller solution of

$$\bar{t}z^2 + (p - 1)z + t = 0$$

that is, the smaller of

$$z = \frac{1 - p \pm \sqrt{(p-1)^2 - 4|t|^2}}{2\bar{t}}.$$

Noting that $-1 < p < 1$ and $(p-1)^2 \geq \left(1 - (1-|t|)^2 + |t|^2\right)^2 = 4|t|^2$, we see that the numerator is a positive number in either case, so the Denjoy-Wolff point is

$$\hat{\alpha} = \frac{1 - p - \sqrt{(p-1)^2 - 4|t|^2}}{2\bar{t}}.$$

As before, $\lim_{n\to\infty} f \circ \psi_n = f(\hat{\alpha})$, and $\|C_\varphi\|^2 \leq |f(\hat{\alpha})|$. On the other hand, since

$$\begin{aligned} C_\varphi^* C_\varphi(K_{\hat{\alpha}}) &= (C_\varphi^* C_\varphi)^*(K_{\hat{\alpha}}) = (T_f C_\psi)^*(K_{\hat{\alpha}}) \\ &= C_\psi^* T_f^*(K_{\hat{\alpha}}) = \overline{f(\hat{\alpha})} K_{\psi(\hat{\alpha})} = \overline{f(\hat{\alpha})} K_{\hat{\alpha}}, \end{aligned}$$

we see that $\|C_\varphi\|^2 = \|C_\varphi^* C_\varphi\| \geq |f(\hat{\alpha})|$. Therefore, in this case, $\|C_\varphi\| = \sqrt{|f(\hat{\alpha})|}$ which is the conclusion of the theorem. ■

Except when $s > 0$ and $|t| = 1 - s$, the operator ${C_\varphi}^2$ is compact and the spectral radius is 1. Except when $t = 0$, the norm of C_φ is greater than 1, so in general, we see the spectral radius is less than the norm.

Co-subnormality for the Inner Hyperbolic Transformations

Let φ be an inner linear fractional transformation with fixed points ± 1. Nordgren, Rosenthal , and Wintrobe [14] and Cowen and Kriete [5] have given proofs that such C_φ^* are subnormal. Nordgren, Rosenthal, and Wintrobe, in addition, study other properties of these operators, and Cowen and Kriete compute the associated measure and study co-subnormality of other composition operators. In this section, we use the adjoint calculation to give a very easy proof of the co-subnormality based on a condition of Embry [8]. This proof does not construct the associated scalar measure, but constructs an associated operator measure.

THEOREM 4. *For $0 < r$, $r \neq 1$, if*

$$\varphi(z) = \frac{(r^{-1} + r)z + (r^{-1} - r)}{(r^{-1} - r)z + (r^{-1} + r)}$$

is the associated inner linear fractional transformation with fixed points ± 1, then C_φ^* is subnormal.

PROOF. We will use Embry's condition [8]:
S is subnormal if and only if there is a positive operator measure Q such that

$$(S^*)^n S^n = \int t^{2n} dQ(t).$$

To apply Embry's condition to C_φ^*, we must calculate ${C_\varphi}^n (C_\varphi^*)^n = C_{\varphi_n} C_{\varphi_n}^*$.

A straightforward calculation gives

$$\varphi_n(z) = \frac{\frac{1}{2}(r^{-n} + r^n)z + \frac{1}{2}(r^{-n} - r^n)}{\frac{1}{2}(r^{-n} - r^n)z + \frac{1}{2}(r^{-n} + r^n)}.$$

By theorem 2, $C_\varphi^* = T_g C_\sigma T_h^*$ where

$$g(z) = 2\left(-(r^{-1} - r)z + (r^{-1} + r)\right)^{-1},$$

$$\sigma(z) = \frac{(r^{-1} + r)z - (r^{-1} - r)}{-(r^{-1} - r)z + (r^{-1} + r)},$$

$$\text{and } h(z) = \frac{1}{2}\left((r^{-1}-r)z + (r^{-1}+r)\right).$$

Since $\sigma(z) = \varphi_{-1}$ and $g \circ \varphi = h$, we have

$$C_\varphi C_\varphi^* = C_\varphi T_g C_\sigma T_h^* = T_{g\circ\varphi} C_{\sigma\circ\varphi} T_h^* = T_h T_h^*.$$

This means

$$C_{\varphi_n} C_{\varphi_n}^* = T_{(\frac{1}{2r^n}-\frac{r^n}{2})z+\frac{1}{2r^n}+\frac{r^n}{2}} T_{(\frac{1}{2r^n}-\frac{r^n}{2})\bar{z}+\frac{1}{2r^n}+\frac{r^n}{2}}$$

$$= \left(\frac{1}{4r^{2n}} - \frac{1}{2} + \frac{r^{2n}}{4}\right) T_z T_{\overline{z}} + \left(\frac{1}{4r^{2n}} - \frac{r^{2n}}{4}\right)(T_z + T_{\overline{z}}) + \left(\frac{1}{4r^{2n}} + \frac{1}{2} + \frac{r^{2n}}{4}\right) I$$

$$= \frac{1}{r^{2n}}\left(\frac{1}{4}(T_z T_{\overline{z}} + T_z + T_{\overline{z}} + I)\right) + \frac{1}{2}(I - T_z T_{\overline{z}}) + r^{2n}\left(\frac{1}{4}(T_z T_{\overline{z}} - T_z - T_{\overline{z}} + I)\right)$$

$$= \frac{1}{r^{2n}}\left(\frac{1}{4}(T_z+I)(T_z+I)^*\right) + \frac{1}{2}\left(I - T_z T_z^*\right) + r^{2n}\left(\frac{1}{4}(T_z-I)(T_z-I)^*\right)$$

$$= \int_0^r t^{2n} dQ(t)$$

where

$$\begin{aligned} Q(\{r^{-1}\}) &= \frac{1}{4}(T_z+I)(T_z+I)^*, \\ Q(\{1\}) &= \frac{1}{2}\left(I - T_z T_z^*\right), \\ \text{and } Q(\{r\}) &= \frac{1}{4}(T_z-I)(T_z-I)^*. \end{aligned}$$

Thus Q is a positive operator valued measure, and Embry's theorem implies that C_φ^* is subnormal. ■

COROLLARY *For $0 < r$, $r \neq 1$, let*

$$\varphi_t(z) = \frac{(r^{-t}+r^t)z + (r^{-t}-r^t)}{(r^{-t}-r^t)z + (r^{-t}+r^t)}$$

be the associated representation of the group of inner linear fractional transformations with fixed points ± 1. Then the group of operators $\{C_{\varphi_t}^* : t \in \mathbb{R}\}$ is a subnormal group.

PROOF. By a theorem of Ito [9], it is sufficient to prove that each operator $C_{\varphi_t}^*$ is subnormal which is a consequence of theorem 4. ■

Hyponormality and Subnormality of C_φ

In [5, theorem 1.2], it is noted that if C_φ is hyponormal, then $\varphi(0) = 0$. In this section, using the adjoint formula of theorem 2, we find all hyponormal composition operators with linear fractional symbol. As Nordgren observed in [12], if φ is any inner function with $\varphi(0) = 0$, then C_φ is an isometry and is subnormal, so we have not found a complete list of all hyponormal composition operators on H^2.

An easy calculation shows that if $\varphi(z) = z(uz+v)^{-1}$ then φ maps D into itself if and only if $|v| \geq 1 + |u|$. H. J. Schwartz [15] proved that C_φ is normal whenever $u = 0$, so the following theorem covers the remaining cases.

THEOREM 5. *For* $u \neq 0$ *and* $|v| \geq 1 + |u|$, *if* $\varphi(z) = z(uz + v)^{-1}$, *the following are equivalent:*

(i) C_φ *is subnormal.*

(ii) C_φ *is hyponormal.*

(iii) $v > 1$ *and* $|u| = v - 1$.

PROOF. Subnormality always implies hyponormality, so $(i) \Rightarrow (ii)$ trivially.

We begin by noting that the vector $K_0 = 1$ is an eigenvector for both C_φ and C_φ^* since $(C_\varphi 1)(z) = 1(\varphi(z)) = 1$ and $C_\varphi^*(K_0) = K_{\varphi(0)} = K_0$. This means that zH^2, which is the orthogonal complement of the subspace of constants, is a reducing subspace for C_φ. Therefore C_φ is hyponormal or subnormal if and only if $C_\varphi|_{zH^2}$ is hyponormal or subnormal.

Choosing $\alpha^{-2} = v$ and $\beta = \alpha u$, a normalized form for φ is

$$\varphi(z) = \alpha z(\beta z + \alpha^{-1})^{-1}.$$

By theorem 2, this means $C_\varphi^* = T_g C_\sigma T_h^*$ where

$$g(z) = (\overline{\alpha}^{-1} - 0z)^{-1} = \overline{\alpha},$$
$$\sigma(z) = (\overline{\alpha}z - \overline{\beta})(\overline{\alpha}^{-1} - 0z)^{-1} = \overline{v}^{-1}z - \overline{(u/v)},$$
$$\text{and } h(z) = \alpha^{-1} + \beta z.$$

Thus

$$C_\varphi^* = \overline{\alpha} C_\sigma T_{\overline{\alpha}^{-1} + \overline{\beta z}} = C_\sigma T_{1 + \overline{(u/v)z}}.$$

If zF is in zH^2, then this formula shows that

$$\begin{aligned} C_\varphi^*(zF) &= C_\sigma T_{1+\overline{(u/v)z}}(zF) \\ &= C_\sigma \left(zF + \overline{(u/v)}F\right) \\ &= \left(\overline{v}^{-1}z - \overline{(u/v)}\right) F \circ \sigma + \overline{(u/v)} F \circ \sigma \\ &= \overline{v}^{-1} zF \circ \sigma = z(\overline{v}^{-1} F \circ \sigma). \end{aligned}$$

That is, if U is the unitary operator from H^2 onto zH^2 given by $UF = zF$, then

$$U^*(C_\varphi^*|_{zH^2})U = \overline{v}^{-1} C_\sigma.$$

Taking adjoints in this equation, we see that C_φ is subnormal or hyponormal if and only if C_σ^* is.

If (iii) is true, then $0 < v^{-1} < 1$ and $|u|/v = 1 - v^{-1}$ so by [5, theorem 2.2] C_σ^* is subnormal and C_φ is also.

On the other hand, if C_φ is hyponormal, then C_σ^* is hyponormal and its spectral radius and norm are equal. By [3, theorem 2.1] the spectral radius of C_σ is 1 if σ has a fixed point in D. Since $\sigma(0) = -\overline{(u/v)} \neq 0$, the norm of C_σ is not 1 and σ does not have a fixed point in D. It follows that σ has a fixed point c on the unit circle, and by [3, theorem 2.1], we see that $0 < \sigma'(c) = \overline{v}^{-1}$, and the spectral radius of C_σ is $\sqrt{v}$. Now since $\|C_\sigma\| = \sqrt{v}$, theorem 3 implies that $|u|/v = 1 - v^{-1}$ so that $|u| = v - 1$ and $v > 1$ as in (iii). ∎

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