Hyponormal and Subnormal Toeplitz Operators

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This paper is my view of the past, present, and future of Problem 5 of Halmos's 1970 lectures "Ten Problems in Hilbert Space " [12] (see also [13]):

Is every subnormal Toeplitz operator either normal or analytic?

We recall that for φ in $L^\infty(\partial D)$, the *Toeplitz operator* T_φ is the operator on the Hardy space H^2 of the unit disk D, given by $T_\varphi h = P\varphi h$ where h is in H^2 and P is the orthogonal projection of $L^2(\partial D)$ onto H^2. An operator S on a Hilbert space $\mathcal{H}$ is *subnormal* if there is a normal operator N on $\mathcal{K} \supset \mathcal{H}$ such that $\mathcal{H}$ is invariant for N and $N|_{\mathcal{H}} = S$.

The question is natural because the two classes, the normal and analytic Toeplitz operators, are fairly well understood and are obviously subnormal. The normal Toeplitz operators were characterized by Brown and Halmos in 1964.

Theorem 1 ([4], page 98) *The Toeplitz operator T_φ is normal if and only if $\varphi = \alpha + \beta\rho$ where α and β are complex numbers and ρ, in L^∞, is real valued.*

A Toeplitz operator T_φ is called *analytic* if φ is in H^∞, that is, φ is a bounded analytic function on D. These are easily seen to be subnormal: $T_\varphi h = P\varphi h = \varphi h = L_\varphi h$ for h in H^2, where L_φ is the normal operator of multiplication by φ on $L^2(\partial D)$.

All progress on this question has begun with the study of the self-commutator of T_φ. A subnormal operator S is *hyponormal*, that is, its self-commutator, $S^*S - SS^*$, is positive. It is not difficult to show that the range of the self-commutator of a subnormal operator is an invariant subspace of S^* ([20], Theorem 5).

Halmos almost certainly believed the answer to his question would be yes, and his intuition was soon bolstered as several results appeared that showed the answer is yes for certain classes. I'll prove the first of these, a

1972 theorem of Ito and Wong, because its proof is typical in its use of the self-commutator. An *inner function* is a function in H^∞ that has modulus 1 almost everywhere on the unit circle.

Theorem 2 ([14], Theorem 1). *If φ is a polynomial in χ and $\bar{\chi}$, where χ is an inner function, then T_φ is subnormal if and only if it is normal or analytic.*

Proof. Since $T_{f\circ\chi}$ is unitarily equivalent to an inflation of T_f for any f in L^∞, it is sufficient to prove the result for $\chi(z) = z$ ([8], Theorem 1).

Suppose T_φ is subnormal but not analytic, say

$$\varphi = a_{-n}\bar{z}^n + \cdots + a_0 + \cdots + a_m z^m$$

where $n > 0$ and $a_{-n} \neq 0$. Let $C = T_\varphi^* T_\varphi - T_\varphi T_\varphi^*$, the self-commutator of T_φ.

If $j \geq k = \max\{m, n\}$, then, $z^j\varphi$ and $z^j\bar{\varphi}$ are both analytic so $z^k H^2 \subset$ kernel C. That is, range C is contained in span $\{1, z, \ldots, z^{k-1}\}$.

If C were not 0, then we could choose f of maximal degree in range C, say, $f = b_0 + \cdots + b_l z^l$, where $b_l \neq 0$. But since T_φ is subnormal, range C is invariant for T_φ^* and $T_\varphi^* f = \cdots + \bar{a}_{-n} b_l z^{n+l}$ which would have higher degree than f. Thus $C = 0$ and T_φ is normal.

■

Ito and Wong were apparently the first to consider hyponormal Toeplitz operators. In the same paper ([14], page 158), they gave the interesting example $T_{z+\frac{1}{2}\bar{z}}$. This was the first *easy* example of a hyponormal operator that is not subnormal. Their example generalizes.

Proposition 3 *Let A be hyponormal. Then $A + \lambda A^*$ is hyponormal if and only if $|\lambda| \leq 1$.*

Proof.

$$(A + \lambda A^*)^*(A + \lambda A^*) - (A + \lambda A^*)(A + \lambda A^*)^* = (1 - |\lambda|^2)(A^*A - AA^*).$$

■

Later, Amemiya, Ito, and Wong showed by a similar argument that the answer to Halmos's question is yes if T_φ is quasinormal.

Theorem 4 [2] *If T_φ commutes with $T_\varphi^* T_\varphi$ then either T_φ is normal or φ is analytic and $\varphi = \lambda\chi$, where χ is an inner function.*

Corollary *If T_φ is subnormal with rank 1 self-commutator, then $\varphi = \alpha + \beta\chi$ where α and β are numbers and χ is an inner linear fractional transformation.*

(This corollary also follows from a more general theorem of B. Morrel [18], page 508.)

The deepest work in this direction is that of Abrahamse, for which we need the following definition.

Definition ([16], page 187) A function φ in $L^\infty(\partial D)$ is of *bounded type* (or in the *Nevanlinna class*) if there are functions ψ_1, ψ_2 in $H^\infty(D)$ such that $\varphi(e^{i\theta}) = \frac{\psi_1(e^{i\theta})}{\psi_2(e^{i\theta})}$ for almost all θ in ∂D.

Clearly, rational functions in $L^\infty(\partial D)$ are of bounded type: they are quotients of analytic polynomials. A polynomial p in z and $\bar{z}$ is of bounded type because on the unit circle, $p(z, \bar{z}) = p(z, z^{-1})$ which is a rational function.

Theorem 5 [1]
If (1) T_φ is hyponormal,
(2) φ or $\bar{\varphi}$ is of bounded type,
and (3) $ker(T_\varphi^* T_\varphi - T_\varphi T_\varphi^*)$ is invariant for T_φ,
then T_φ is normal or analytic.

Since (3) holds for any subnormal operator ([20], Theorem 5), Abrahamse obtains the conclusion for subnormals.

Corollary *If T_φ is subnormal and φ or $\bar{\varphi}$ is of bounded type, then T_φ is normal or analytic.*

Abrahamse concluded his paper by asking several questions, including the following:

> Which hyponormal weighted shifts are unitarily equivalent to Toeplitz operators ?
>
> Is the Bergman shift unitarily equivalent to a Toeplitz operator?

Recall that an operator W is a (unilateral) *weighted shift* if there is an orthonormal basis $e_0, e_1, \ldots$ and weights $w_n > 0$ such that $We_j = w_j e_{j+1}$ for $j = 0, 1, 2, \ldots$. An easy calculation shows W is hyponormal if and only if $w_0 \le w_1 \le w_2 \le \cdots$. It is more difficult to show that W is subnormal if and only if

$$(w_0 w_1 \cdots w_{k-1})^2 = \int t^{2k} d\nu(t)$$

for some probability measure ν with support in $[0, \|W\|]$ (due independently to Berger and to Gellar and Wallen [7], page 159). The *Bergman shift* is the subnormal shift with weights $w_n^2 = (n+1)(n+2)^{-1}$ for $n = 0, 1, 2, \cdots$.

Sun Shunhua proved the following remarkable theorem by carefully examining the action of the self-commutator on e_0 in the shifted basis.

Theorem 6 [22] *If T_φ is a hyponormal weighted shift, then there is a number α, $0 \le \alpha \le 1$ so that the weights are $w_n^2 = (1 - \alpha^{2n+2})\|T_\varphi\|^2$.*

The case $\alpha = 0$ is T_z. Sun Shunhua left open the question of existence for the cases $\alpha > 0$, but this did answer Abrahamse's second question.

Corollary *The Bergman shift is not unitarily equivalent to a Toeplitz operator.*

Proof.

$$\frac{n+1}{n+2} \ne 1 - \alpha^{2n+2} \text{ for any } \alpha > 0$$

■

Examination of the proof of the theorem reveals that such a φ must have $\psi = \varphi - \alpha\bar{\varphi}$ in H^∞. Moreover, the matrix for T_ψ in the shifted basis is a compact perturbation of

$$\begin{pmatrix} 0 & -\alpha & 0 & 0 & \\ 1 & 0 & -\alpha & 0 & \\ 0 & 1 & 0 & -\alpha & \\ 0 & 0 & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

which is the matrix for $T_{z-\alpha\bar{z}}$ in the usual basis. It follows that the Fredholm indices and the essential spectra of these Toeplitz operators must be the same. Since T_ψ is an analytic Toeplitz operator, ψ must be a conformal mapping of D onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$, and $\varphi = (1-\alpha^2)^{-1}(\psi + \alpha\bar{\psi})$. Knowing that this would have to be the symbol made it possible to verify that this works.

Theorem 7 [9] *Let $0 < \alpha < 1$ be given and let ψ be a conformal map of the disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. If $\varphi = (1-\alpha^2)^{-1}(\psi + \alpha\bar{\psi})$, then T_φ is a weighted shift with weight sequence $w_n^2 = 1 - \alpha^{2n+2}$ and is subnormal but neither normal nor analytic.*

The next step forward was Sun Shunhua's characterization of those Toeplitz operators such that $T_\varphi + T_\varphi^*$ and $T_\varphi T_\varphi^*$ commute, what Campbell called the Θ-class. Surprisingly, the answer again involved these ellipse maps: $\varphi = \psi + \beta\bar{\psi}$ gives rise to a Θ-class Toeplitz operator for a certain choice of β. Moreover, since $|\beta| < 1$, the operator T_φ is hyponormal and a theorem of Campbell, [5], implies T_φ is subnormal.

Thus, at this point, we know that $T_{\psi+0\bar{\psi}}$ (which is analytic), $T_{\psi+\alpha\bar{\psi}}$, $T_{\psi+\beta\bar{\psi}}$, and $T_{\psi+1\bar{\psi}}$ (which is normal) are all subnormal, and from the generalization of Ito and Wong's observation, $T_{\psi+\lambda\bar{\psi}}$ is hyponormal for $|\lambda| \leq 1$. This strongly suggests the question:

For which λ is $T_{\psi+\lambda\bar{\psi}}$ subnormal?

The answer to this question is provided by the following recent theorem.

Theorem 8 ([10], Theorem 2.4) *Let λ be a complex number, let $0 < \alpha < 1$, and let ψ be the conformal map of the disk onto the interior of the ellipse with vertices $\pm i(1+\alpha)$ passing through $\pm(1-\alpha)$ where $0 < \alpha < 1$. For $\varphi = \psi + \lambda\bar{\psi}$, the Toeplitz operator T_φ is subnormal if and only if $\lambda = \alpha$ or $\lambda = (\alpha^k e^{i\theta} + \alpha)(1 + \alpha^{k+1}e^{i\theta})^{-1}$ for some $k = 0, 1, 2, \ldots$ and $0 \leq \theta < 2\pi$.*

Note that $k = 0$ in the theorem means $|\lambda| = 1$, that is, $T_{\psi+\lambda\bar{\psi}}$ is normal; $\lambda = \alpha$ (which corresponds to $k = \infty$) is the weighted shift case; $k = 1, \theta = \pi$ is the analytic case $T_{\psi+0\bar{\psi}}$; and the Θ-class case of Sun Shunhua corresponds to $k = 2$.

This theorem follows by simple algebra from the Cowen-Long result and the following result on the special weighted shifts that are multiples of the Toeplitz operators in Theorem 7.

Proposition 9 [10] *Let T be the weighted shift with weights*

$$w_n^2 = \sum_{j=0}^{n} \alpha^{2j}.$$

Then $T + \mu T^*$ is subnormal if and only if $\mu = 0$ or $|\mu| = \alpha^k$ for $k = 0, 1, 2, \ldots$.

The proof of the proposition is a modification of the matrix constructions of Ando [3] and Stampfli [21] for the minimal normal extension of a subnormal operator. The heart of the matter is: If S is subnormal and $\begin{pmatrix} S & B \\ 0 & A \end{pmatrix}$ is its minimal normal extension, what choice do we have for B? The answer: We may replace B by $\tilde{B}$ as long as $BB^* = \tilde{B}\tilde{B}^*$ and their kernels have the same dimension. In particular, by polar factorization, we may replace B by a non-negative operator.

Sketch of proof. Let D be the diagonal operator whose k-th diagonal entry is α^k. Then, D is positive and $D^2 = T^*T - TT^*$. Using rotations and the fact that T is subnormal and $T + T^*$ is normal, it is sufficient to give the proof for $\mu = s$, where $0 < s < 1$. Let $A_0 = T + sT^*$.

Suppose A_0 is subnormal, and $\begin{pmatrix} A_0 & X_1 \\ 0 & Y_1 \end{pmatrix}$ is its minimal normal extension. The normality of this matrix implies $X_1X_1^* = A_0^*A_0 - A_0A_0^* = (1 - s^2)D^2$. A theorem of Olin ([17], page 228) implies that either X_1 has kernel (0) or X_1 has infinite-dimensional kernel. Letting $B_1 = r_1D$ where $r_1^2 = (1 - s^2)$, it follows that the minimal normal extension has a representation as either

$$\begin{pmatrix} A_0 & B_1 \\ 0 & A_1 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} A_0 & B_1 & 0 \\ 0 & A_1 & X_2 \\ 0 & 0 & Y_2 \end{pmatrix}.$$

In either case, normality implies that $A_1 = \alpha T + s\alpha^{-1}T^*$. In the former case, we also get $A_1^*A_1 - A_1A_1^* = -B_1B_1^*$ which means $\alpha = s$. In the latter case, using a positivity condition that is a consequence of the normality, we find that $\alpha < s < 1$ is impossible.

After $n - 1$ such steps, if $s \neq \alpha^k$ for $k = 1, 2, \ldots, n - 1$, we have shown that $s < \alpha^{n-1}$ and that, for $A_k = \alpha^kT + s\alpha^{-k}T^*$ and $B_k = r_kD$, the minimal normal extension is unitarily equivalent either to

$$\begin{pmatrix} A_0 & B_1 & \cdots & 0 \\ 0 & A_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}$$

in which case normality implies $s = \alpha^n$ or to

$$\begin{pmatrix} A_0 & B_1 & \cdots & 0 & 0 \\ 0 & A_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_n & X_{n+1} \\ 0 & 0 & \cdots & 0 & Y_{n+1} \end{pmatrix}$$

and normality implies that $\alpha^n < s < \alpha^{n-1}$ is impossible. ■

A miracle has occurred: all the entries in the matrix model were explicitly computable for this weighted shift!

Standard facts about unitary equivalence of shifts and Theorem 7 give some non-obvious unitary equivalences of Toeplitz operators.

Corollary [10] *The analytic Toeplitz operator T_ψ is unitarily equivalent to each of the non-analytic Toeplitz operators T_φ with*

$$\varphi = ie^{-i\theta/2}(1-\alpha^2)^{-1}\left[(1+\alpha^2 e^{i\theta})\psi + \alpha(1+e^{i\theta})\bar{\psi}\right]$$

for $-\pi < \theta < \pi$.

Proof. Since T is a weighted shift, $T + sT^*$ is unitarily equivalent to $T + \lambda T^*$ whenever $|\lambda| = s$. Rewriting this unitary equivalence in terms of the Toeplitz operator T_ψ gives the result. ■

Now Theorem 7 is interesting and surprising, but it is not very encouraging. It says that the subnormality of these Toeplitz operators depends on some combinatorial coincidences and suggests that subnormality of Toeplitz operators may be the wrong question to be studying. Perhaps more progress can be made studying the hyponormality of Toeplitz operators.

Very little work has gone into discovering which symbols in L^∞ give hyponormal Toeplitz operators and into developing an adequate theory for them. Much of what is known is in the form of "folk theorems". I will give some elementary observations and suggest some questions for further study. The intuition is that T_φ is hyponormal if the analytic part of φ dominates the conjugate analytic part. Some of the results below may be interpreted as making a precise statement supporting this intuition. For example, in the following unpublished proposition of Wogen, we feel $\chi\varphi$ is the same size as φ but is "more analytic".

Proposition 10 [25] *If χ is inner and φ in L^∞ is such that T_φ is hyponormal, then $T_{\chi\varphi}$ is also hyponormal.*

Proof. For h in H^2, since T_χ is an isometry and T_φ is hyponormal, we have

$$\|T_{\chi\varphi}h\| \geq \|T_{\bar{\chi}}T_{\chi\varphi}h\| = \|T_\varphi h\| \geq \|T_\varphi^* h\| = \|T_{\bar{\varphi}}h\| \geq \|T_{\bar{\chi}}T_{\bar{\varphi}}h\| = \|T_{\chi\varphi}^* h\|.$$

Thus, $T_{\chi\varphi}$ is also hyponormal. ∎

We recall the definition of Hankel operators (see for example [19]). As usual, P is the projection from L^2 onto H^2.

Definition For φ in L^∞ the Hankel operator $H_\varphi : H^2 \to (H^2)^\perp$ is given by $H_\varphi h = (I-P)\varphi h$, for h in H^2.

The following proposition is not completely general because not all φ can be split as in the hypothesis. With care, regarding the Toeplitz operators as unbounded operators, it can be improved to general L^∞ functions.

Proposition 11 *Suppose f and g are in H^∞ and suppose $\varphi = f + \bar{g}$. The following are equivalent.*

(1) T_φ is hyponormal.
(2) For every h in H^2 $\|fh\|^2 - \|P\bar{f}h\|^2 \geq \|gh\|^2 - \|P\bar{g}h\|^2$.
(3) $T_f^*T_f - T_fT_f^* \geq T_g^*T_g - T_gT_g^*$.
(4) $H_{\bar{f}}^*H_{\bar{f}} \geq H_{\bar{g}}^*H_{\bar{g}}$.
(5) For every h in H^2, we have $\|H_{\bar{f}}h\| \geq \|H_{\bar{g}}h\|$.

Proof. Compute. ∎

Corollary *Let χ be inner, and let F and G be in H^∞ with $f = \chi F$ and $g = \chi G$. If $T_{f+\bar{g}}$ is hyponormal, then $T_{F+\bar{G}}$ is also hyponormal.*

Proof. Using (2) above, we have

$$\|Fh\|^2 - \|P\bar{F}h\|^2 = \|f(\chi h)\|^2 - \|P\bar{f}(\chi h)\|^2$$
$$\geq \|g(\chi h)\|^2 - \|P\bar{g}(\chi h)\|^2 = \|Gh\|^2 - \|P\bar{G}h\|^2.$$
∎

Using (4) and the fact that for χ inner, $H_{\bar{\chi}}^* H_{\bar{\chi}}$ is the projection of H^2 onto $(\chi H^2)^\perp$, we get the following corollary.

Corollary *Let χ be inner and g be in H^∞. Then $T_{\chi+\bar{g}}$ is hyponormal if and only if g is in $(z\chi H^2)^\perp$ and $\|H_{\bar{g}}\| \leq 1$.*

If χ is a finite Blaschke product, then $(z\chi H^2)^\perp$ is finite dimensional and for each g in $(z\chi H^2)^\perp$, $H_{\bar{g}}$ is finite rank, so finding the set of g such that $T_{\chi+\bar{g}}$ is hyponormal is a computation in finite dimensional linear algebra. For example, it is tedious but not difficult to verify that $T_{z^2+\bar{g}}$ is hyponormal if and only if $g = \alpha + \beta z + \gamma z^2$ where $|\beta| + |\gamma|^2 \leq 1$.

We are interested more generally in the set of g such that $T_{f+\bar{g}}$ is hyponormal for a given f. This motivates the following definition. To avoid difficulties with splitting functions in L^∞, we formulate the definition for f and g in H^2 without regard to whether they are the analytic parts of functions in L^∞.

Definition Let $\mathcal{H} = \{h \in H^\infty \ : h(0) = 0 \text{ and } \|h\|_2 \leq 1\}$. For f in H^2, let G_f denote the set of g in H^2 such that for every h in H^2,

$$\sup_{h_0 \in \mathcal{H}} |\langle hh_0, f\rangle| \geq \sup_{h_0 \in \mathcal{H}} |\langle hh_0, g\rangle|$$

Note that in the definition of $\mathcal{H}$, we have used the H^2 norm of the H^∞ function h, and that $\mathcal{H}$ is dense in the unit ball of zH^2. To see how this definition is relevant to our work, suppose p is in H^∞ and h is in H^2. Then we have

$$\sup_{h_0 \in \mathcal{H}} |\langle hh_0, p\rangle| = \sup_{h_0 \in \mathcal{H}} |\langle \bar{p}h, \bar{h}_0\rangle| = \sup_{h_0 \in \mathcal{H}} |\langle (I-P)\bar{p}h, \bar{h}_0\rangle| = \|H_{\bar{p}}h\|.$$

Thus, when f and g are both in H^∞, this means, by (5) of proposition 11, that $T_{f+\bar{g}}$ is hyponormal if and only if g is in G_f. The following theorem gives some elementary properties of G_f.

Theorem 12 *For f in H^2, the following hold.*

(0) $G_f = G_{f+\lambda}$ for all complex numbers λ.

(1) f is in G_f.

(2) If g is in G_f, then $g+\lambda$ is in G_f for all complex numbers λ.

(3) G_f is balanced and convex; that is, if g_1 and g_2 are in G_f and $|s|+|t|\le 1$, then sg_1+tg_2 is also in G_f.

(4) G_f is weakly closed.

(5) If χ is inner and χg is in G_f, then g is in G_f.

Proof. Properties (0), (1), and (2) are obvious.
(3) For h in H^2, we have

$$\sup_{h_0\in\mathcal{H}} |<hh_0, f>| \ge |s| \sup_{h_0\in\mathcal{H}} |<hh_0, f>| + |t| \sup_{h_0\in\mathcal{H}} |<hh_0, f>|$$

$$\ge |s| \sup_{h_0\in\mathcal{H}} |<hh_0, g_1>| + |t| \sup_{h_0\in\mathcal{H}} |<hh_0, g_2>|$$

$$\ge \sup_{h_0\in\mathcal{H}} |<hh_0, sg_1+tg_2>|.$$

(4) Let h be in H^2. Suppose g_α is in G_f for each α in a directed set and suppose $g_\alpha \to g$ weakly. Fix h_1 in $\mathcal{H}$. Thus,

$$|<hh_1, g>| = \lim_\alpha |<hh_1, g_\alpha>| \le |<hh_1, f>| \le \sup_{h_0\in\mathcal{H}} |<hh_0, f>|.$$

Since this is true for all such h_1, we have g in G_f.
(5) Let h be in H^2. Since $h_0\in\mathcal{H}$ implies $\chi h_0\in\mathcal{H}$, for χg in G_f, we have

$$\sup_{h_0\in\mathcal{H}} |<hh_0, f>| \ge \sup_{h_0\in\mathcal{H}} |<hh_0, \chi g>|$$

$$\ge \sup_{h_0\in\mathcal{H}} |<h(\chi h_0), \chi g>| = \sup_{h_0\in\mathcal{H}} |<hh_0, g>|,$$

so g is in G_f.

■

Corollary $T_z^* G_f \subset G_f$.

Proof. By (2) g in G_f implies $g-g(0)$ is in G_f and (5) implies $T_z^* g = (g-g(0))/z$ is in G_f.

■

I am proposing the study of the hyponormal Toeplitz operators; the study of the properties of G_f seems like a reasonable place to start. Several questions seem interesting.

Question 1 *Can G_f be characterized? In particular, do (1) to (5) of Theorem 12 characterize G_f?*

For $f = z^2$, these properties determine G_f, and the computations are easier than the norm computations referred to earlier: By (1) and (2), $z^2 - a^2 \in G_{z^2}$ for all a in the unit disk, so (5) implies

$$-\bar{a}z^2 + (1 - |a|^2)z + a = \left(\frac{z-a}{1-\bar{a}z}\right)^{-1} (z^2 - a^2) \text{ is in } G_{z^2}.$$

Using (2) and (3) we see that $G_{z^2} \supset \{g(z) = \beta z + \gamma z^2 : |\beta| + |\gamma|^2 = 1\}$, and finally that $G_{z^2} \supset \{g(z) = \alpha + \beta z + \gamma z^2 : |\beta| + |\gamma|^2 \leq 1\}$.

Translation by scalars was used to advantage in the above computation, but it may not always be helpful. It may be more convenient to work with the set

$$G'_f = \{g \in G_f : g(0) = 0\}$$

since if g is in G'_f (taking $h \equiv 1$ in the definition of G_f) we have

$$\|g\|_2 = \sup_{h_0 \in \mathcal{H}} | \langle h_0, g \rangle | \leq \sup_{h_0 \in \mathcal{H}} | \langle h_0, f \rangle | = \|f\|_2.$$

This means that G'_f is convex and weakly compact.

Question 2 *What are the extreme points of G'_f? In particular, if $g \in G'_f$ but $\lambda g \notin G'_f$ for $|\lambda| > 1$, is g an extreme point of G'_f?*

A function g is an extreme point of G'_{z^2} if and only if $g(z) = \beta z + \gamma z^2$ where $|\beta| + |\gamma|^2 = 1$, but of course we want a description of the extreme points of G'_f that can be used to compute G'_f, not vice versa. The set of extreme points came up naturally in the computation of G_{z^2} using the properties (1) to (5) above; perhaps it is always so. For the special subset in the question to be the set of extreme points, would require a certain rotundity of the set G'_f. The corollary of Theorem 11 characterizes this subset for f inner, and for $f = z^2$, all are extreme points.

In the study of subnormal Toeplitz operators, these results suggest two more questions.

Question 3 *For which f in H^∞ is there λ, $0 < \lambda < 1$ with $T_{f+\lambda\bar{f}}$ subnormal?*

Question 4 *Suppose ψ is as in Theorem 7. Are there g in G_ψ, $g \neq \lambda\psi + c$, such that $T_{\psi+\bar{g}}$ is subnormal?*

Note that Abrahamse's work [1] relates directly to questions 3 and 4. If $\varphi = f + \bar{g}$ with f, g, in H^∞, then φ is of bounded type if and only if $\bar{g}$ is of bounded type. Thus T_φ subnormal, neither normal nor analytic, implies that $\bar{f}$ and $\bar{g}$ are not of bounded type.

Although isolated results and examples have appeared in the literature, hyponormality of Toeplitz operators has not been systematically studied. I believe that the study of hyponormal Toeplitz operators is of more importance to the understanding of Toeplitz operators than is the study of subnormality, and that in any case, more progress needs to be made on the hyponormality questions before substantial progress on the subnormality questions can be made. I hope I have given you some hints on where to start.

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