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Analytic solutions of Böttcher's functional equation in the unit disk

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Suppose f is analytic on the unit disk D, maps D into itself, and has the Taylor series $f(z) = a_k z^k + a_{k+1} z^{k+1} + \cdots$ where $a_k \neq 0$ and $k \geqq 2$. This paper gives necessary and sufficient conditions for the existence of single-valued analytic solutions defined on all of D to Böttcher's functional equation $\sigma \circ f = \sigma^m$. It is easily seen [5] that the only non-zero solutions occur when $k = m$. There is always a solution of the equation $\sigma \circ f = \sigma^k$ that is holomorphic and univalent *in a neighborhood of zero*: see Valiron [9, pp. 124–127]. When σ is a solution of Böttcher's equation, so is σ^n for $n = 2, 3, \ldots$, so our problem is to determine when one of these has a single-valued continuation to all of the disk. We shall see that the existence of such solutions depends on a multiplicity condition on the zeroes of iterates of f, and we determine all solutions when the condition is met. The solutions are computable in the sense that the Taylor coefficients of σ can be obtained recursively.

As usual, from the solutions of this fundamental equation one can obtain information about solutions of the classical functional equations of Abel and Schroeder and about fractional iterates of f. Since these necessitate consideration of multiple-valued functions, we confine our attention on these questions to those f that are real-valued and increasing on $[0, 1)$. We obtain entirely analogous results to those of Szekeres [8], Kuczma [4], and Ger and Smajdor [3]. The additional information obtained here is that the natural fractional iteration semigroup $f_t(x) = F(x, t)$ is real analytic in t as well as x when $f'(x) > 0$ for $0 < x < 1$ and that if $\lim_{x \to 1^-} f(x) = 1$, this semigroup is actually embedded in a continuous group. References to the extensive literature on the subject of iteration may be found in [6]. I would like to thank the referees for several helpful suggestions and for pointing out some relevant references.

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We begin by examining the appropriate notion of multiplicity. For z_0 in D, n a positive integer, let $m(z_0, n)$ be the multiplicity of the zero of f_n at z_0, that is, if $f_n(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \cdots$ then $m(z_0, n) = \min\{j : b_j \neq 0\}$. (Here f_n denotes the nth iterate of f, that is, $f_1 = f$ and $f_{n+1} = f \circ f_n$ for $n = 1, 2, \ldots$.)

Since $m(z_0, n) > 0$ if and only if $f_n(z_0) = 0$, we have $m(z_0, n) = 0$ for all n for all but countably many z_0 in D. Since $f(z) = a_k z^k + \cdots$, we see $m(0, n) = k^n$ for $n = 1, 2, 3, \ldots$. If $f(z_0) = 0$ and $m(z_0, 1) = j$ (so that $f(z) = b_j(z - z_0)^j + \cdots$) then $m(z_0, n) = jk^{n-1}$ (since $f(f(z)) = a_k(b_j(z - z_0)^j + \cdots)^k + \cdots$, etc.).

For f analytic in D, mapping D into itself with $f(z) = a_k z^k + \cdots$ where $a_k \neq 0$ and $k \geqq 2$, we define the *multiplicity set of* f to be the set $Q_f = \{k^{-n} m(z, n) : z \in D, n = 1, 2, 3, \ldots\}$.

As a consequence of the above observations we have the following.

PROPOSITION. *For f as above, $Q_f \supset \{0, 1\}$. Moreover, if $\{z \in D : f_n(z) = 0$ for some $n\}$ is finite, then Q_f is finite.*

We now compute the multiplicity sets for some particular functions.

EXAMPLE 1. Let f be the inner function $f(z) = z^k \exp((z + 1)(z - 1)^{-1})$, where $k \geqq 2$. Since $f_n(z) = 0$ if and only if $z = 0$, we have $m(z, n) = 0$ for $z \neq 0$, so $Q_f = \{0, 1\}$.

EXAMPLE 2. Let $f(z) = 0.3z^2 - 0.6z^3$, then $Q_f = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$. To see this we observe that $f(z) = 0$ if and only if $z = 0$ or $z = \frac{1}{2}$; $f_2(z) = 0$ if and only if $z = 0$, $z = \frac{1}{2}$ or $z = z^* \simeq -0.8$. Since $|f_k(z)| \leqq (0.9)^7 < 0.5$ for $k \geqq 3$ and since $f_2(z) \neq z^*$ for $|z| < 1$, we see that $m(z, n) = 0$ except $z = 0$, $z = \frac{1}{2}$ and $z = z^*$. It is easy to see that $m(0, n) = 2^n$ for all n, $m(\frac{1}{2}, n) = 2^{n-1}$ for all n, and $m(z^*, n) = 2^{n-2}$ for $n \geqq 2$ with $m(z^*, 1) = 0$.

EXAMPLE 3. For $f(z) = 2^{-8} z^3 (z - \frac{1}{2})^7$, since $|f(z)| < \frac{1}{2}$ for $|z| < 1$, we see $Q_f = \{0, 1, \frac{7}{3}\}$.

EXAMPLE 4. Suppose $B(z) = a_k z^k + \cdots$ is a finite Blaschke product of order M, $a_k \neq 0$ with $k \geqq 2$, and $B(z) \neq a_k z^k$. Since B has order M, $B'(z) = 0$ for at most $M - 1$ points in D. Choose ω in D, $\omega \neq 0$ such that $B(\omega) = 0$ and let $r = \max\{|z| : B'(z) = 0\}$. By Schwarz' lemma, $|B(z)| < |z|$ for $0 < |z| < 1$ and the only fixed point in D is 0, so we can find ω^* in D, with $|\omega^*| > r$, and $B_{l-1}(\omega^*) = \omega$. Let $m = m(\omega^*, l) \geqq 1$. Now if $B_n(z) = \omega^*$ then $m(z, n + l) = m$ and $mk^{-n-l} \in Q_B$. Since B_n maps D onto D for every n, we see that $mk^{-n-l} \in Q_B$ for $n = 1, 2, 3, \ldots$. Carrying the ideas slightly further we see that Q_B is a bounded infinite set.

EXAMPLE 5. Let $z_j = 1 - 2^{-j}$ for $j = 1, 2, 3, \ldots$, let $k \geqq 2$ be given and let

$$f(z) = \frac{1}{2}\left[z \prod_{j=1}^{\infty} \left(\frac{z_j - z}{1 - z_j z}\right)^j\right]^k.$$

(The Blaschke condition guarantees the convergence of the product.) Since $f(D) \subset \frac{1}{2}D$ and $m(z_j, 1) = kj$, we have $Q_f = \{0, 1, 2, 3, \ldots\}$.

We now state the main theorem.

THEOREM 1. *Let $f(z) = a_k z^k + \cdots$, where $k \geqq 2$ and $a_k \neq 0$, be analytic in D with $f(D) \subset D$, and let $\lambda \neq 0$ be given. There is σ analytic in D, with $\sigma(D) \subset D$, such that $\sigma(f(z)) = \lambda\sigma(z)^m$ if and only if $m = k$ and there is an integer l such that lQ_f is a subset of the integers. Moreover, in this case, there is a unique solution $\sigma(z) = \beta z^l + \cdots$ for each such l and solution β of the equation*

$$\beta^{k-1} = \lambda^{-1} a_k^l.$$

The first step in the proof is the construction of a solution near zero for $\lambda = 1$ where $\sigma(z) = \beta z + \cdots$. This step does not involve the multiplicity set. The solutions of the general equation are of the form $(\alpha\sigma)^l$ where α is a complex number and l a positive integer. These local solutions have single-valued extensions to all of the disk exactly when lQ_f is a subset of the integers.

Proof. I. Find $\varepsilon > 0$ such that $|z| < \varepsilon$ and $f(z) = 0$ imply $z = 0$. Let $H = \{\omega : \operatorname{Re}\omega > 0\}$. Define $F : H \to H$ to be any branch of

$$F(\omega) = -\log f(\varepsilon e^{-\omega}) + \log \varepsilon.$$

All branches are single-valued since H is simply connected and F is arbitrarily continuable in H. For fixed ω, since $\varepsilon e^{-\omega - 2\pi i t}$ winds around zero once for $0 \leqq t \leqq 1$, the image $f(\varepsilon e^{-\omega - 2\pi i t})$ winds around zero k times, so $F(\omega + 2\pi i) = F(\omega) + 2\pi k i$. (It follows that there are k branches of $\log f(\varepsilon e^{-\omega})$ on H.)

Now

$$\lim_{x \to \infty} \operatorname{Re} \frac{F(x)}{x} = \lim_{x \to \infty} \frac{-\log|f(\varepsilon e^{-x})| + \log \varepsilon}{x}$$

$$= \lim_{x \to \infty} \frac{-\log|a_k \varepsilon^k e^{-kx} + a_{k+1}\varepsilon^{k+1} e^{-(k+1)x} + \cdots|}{x}$$

$$= \lim_{x \to \infty} k - \frac{\log|a_k \varepsilon^k + a_{k+1}\varepsilon^{k+1} e^{-x} + \cdots|}{x} = k.$$

Letting $F_n(1) = x_n + iy_n$, from [7, p. 440], we have $g(\omega) = \lim_{n\to\infty}[(F_n(\omega) - iy_n)/x_n]$ exists and $g(F(\omega)) = \varphi(g(\omega))$ where $\varphi(\omega) = \alpha\omega + i\beta$, for some $\alpha > 0$ and β real.

Now

$$
\begin{aligned}
g(\omega + 2\pi i) &= \lim_{n\to\infty} \frac{F_n(\omega + 2\pi i) - iy_n}{x_n} \\
&= \lim_{n\to\infty} \frac{F_n(\omega) + 2\pi k^n i - iy_n}{x_n} = \lim_{n\to\infty} \frac{F_n(\omega) - iy_n}{x_n} + 2\pi i \frac{k^n}{x_n} \\
&= g(\omega) + 2\pi i \lim_{n\to\infty} \frac{k^n}{x_n}.
\end{aligned}
$$

Thus $\gamma = \lim_{n\to\infty}(k^n/x_n) = (2\pi i)^{-1}[g(\omega + 2\pi i) - g(\omega)]$ exists.

Since g is univalent for ω in $|\arg \omega| < \pi/4$, $|\omega| > \rho$ for ρ sufficiently large [7, Theorem 3, p. 441], we find that $\gamma \neq 0$.

We also have

$$g(F(\omega + 2\pi i)) = \varphi(g(\omega + 2\pi i)) = \alpha g(\omega + 2\pi i) + i\beta = \alpha g(\omega) + \alpha\gamma 2\pi i + i\beta$$

and on the other hand

$$g(F(\omega + 2\pi i)) = g(F(\omega) + 2\pi k i) = g(F(\omega)) + 2\pi k\gamma i = \alpha g(\omega) + 2\pi k\gamma i + i\beta.$$

Since $0 < \gamma < \infty$, this means $\alpha = k$.

Now define A by $A(\omega) = \gamma^{-1}[g(\omega) + \beta(k-1)^{-1}i]$. Thus $A(\omega + 2\pi i) = A(\omega) + 2\pi i$ and $A(F(\omega)) = kA(\omega)$. Defining $\tilde{\sigma}$ by $\tilde{\sigma}(z) = \exp(-A(-\log z))$ for $0 < |z| < \varepsilon$, we obtain from these relations

$$\exp(-A(-\log z + 2\pi i)) = \exp(-A(-\log z) - 2\pi i) = \exp(-A(-\log z))$$

so that $\tilde{\sigma}$ is single-valued and

$$
\begin{aligned}
\tilde{\sigma}(f(z)) &= \exp(-A(-\log(f(z)))) \\
&= \exp(-A(F(-\log z))) = \exp(-kA(-\log z)) \\
&= [\exp(-A(-\log z))]^k = \tilde{\sigma}(z)^k.
\end{aligned}
$$

By Theorem 3 of [7, p. 441], we have $|A(\omega)| \to \infty$ and $\arg \omega^{-1}A(\omega) \to 0$ as $\omega \to \infty$ with $|\text{Im}\, \omega| \leqq \pi$, which means $\text{Re}\, A(\omega) \to \infty$ as $\omega \to \infty$ with $|\text{Im}\, \omega| \leqq \pi$. It follows (by choosing $|\text{Im}(\log z)| \leqq \pi$) that $\lim_{z\to 0} \tilde{\sigma}(z) = 0$, so $\tilde{\sigma}$ has a removable singularity at 0.

Thus $\bar{\sigma}$ is analytic for $|z|<\varepsilon$ and $\bar{\sigma}(z)=0$ if and only if $z=0$, so $\bar{\sigma}(z)=\bar{b}_j z^j+\cdots$ where $j\geqq 1$, $\bar{b}_j\neq 0$. Under these circumstances, each branch of $\bar{\sigma}^{1/j}$ is single-valued and analytic in $|z|<\varepsilon$ and $(\bar{\sigma}^{1/j})'(0)\neq 0$. Choose one branch and denote it $\bar{\sigma}_0$. We have $[\bar{\sigma}_0(f(z))]^j=\bar{\sigma}(f(z))=[\bar{\sigma}(z)]^k=[\bar{\sigma}_0(z)^k]^j$, which means $\bar{\sigma}_0(f(z))=e^{i\theta}\bar{\sigma}_0(z)^k$ for some real θ. Defining $\sigma_0(z)=e^{i\theta(k-1)^{-1}}\bar{\sigma}_0(z)$, we have that σ_0 is analytic, single-valued for $|z|<\varepsilon$, $\sigma_0(0)=0$, $\sigma_0'(0)\neq 0$, and $\sigma_0(f(z))=\sigma_0(z)^k$. That is, σ_0 is a local solution to our functional equation.

II. Suppose l is an integer such that lQ_f is a subset of the integers. For $|z|<\varepsilon$, we define $\sigma(z)=(\sigma_0(z))^l$ so that $\sigma(z)=b_l z^l+\cdots$ and $\sigma(f(z))=\sigma(z)^k$. We want to show that σ has a single-valued extension on all of D.

For each integer n, one of the branches of $[\sigma(f_n(z))]^{1/k^n}$ for $|z|<\varepsilon$ is $\sigma(z)$. Consider the analytic continuation of this branch to $\{z:|f_n(z)|<\varepsilon\}$. The possible branch points of this function are the points z_0 such that $f_n(z_0)=0$. For δ small positive, $\gamma(t)=z_0+\delta e^{2\pi it}$, $0\leq t\leq 1$, winds around z_0 once and $f_n(\gamma(t))$ winds around zero $m(z_0,n)$ times, so $\sigma(f_n(\gamma(t)))$ winds around zero $lm(z_0,n)$ times. By the choice of l, k^n divides $lm(z_0,n)$ so continuing $[\sigma(f_n(z))]^{1/k^n}$ along γ gives the same function element for $t=0$ as for $t=1$. In other words $[\sigma(f_n(z))]^{1/k^n}$ is single-valued near each branch point, so it is a single-valued analytic continuation of $\sigma(z)$ to $\{z:|f_n(z)|<\varepsilon\}$. In particular, this means that, if $m<n$ and $|f_m(z)|<\varepsilon$ then $[\sigma(f_m(z))]^{1/k^m}=[\sigma(f_n(z))]^{1/k^n}$. Thus, defining σ on D by $\sigma(z)=[\sigma(f_n(z))]^{1/k^n}$, where n is an integer such that $|f_n(z)|<\varepsilon$, makes σ into a single-valued analytic function on D with $\sigma(f(z))=\sigma(z)^k$.

III. Now, if $\lambda\neq 0$ and if $\beta^{k-1}=\lambda^{-1}$, then $h=\beta\sigma$ satisfies the equation $h\circ f=\lambda h^k$. This means there are at least $k-1$ distinct solutions of the equation $h\circ f=\lambda h^k$ with $h(z)=c_l z^l+\cdots$ where $c_l\neq 0$. On the other hand, if h is such a solution, equating the coefficients of z^{kl} yields $c_l a_k^l=\lambda c_l^k$ so that $c_l^{k-1}=\lambda^{-1}a_k^l$ and we see that there are at most $k-1$ possible leading coefficients, c_l. Equating coefficients of z^{kl+j} for $j=1,2,\ldots$ we obtain

$$S_j(f,c_l,c_{l+1},\ldots,c_{l+i})=\lambda c_{l+j}c_l^{k-1}+R_j(\lambda,c_l,\ldots,c_{l+j-1})$$

where $i=[j/k]$ and S_j and R_j are functions depending on the variables indicated. Since $k\geqq 2$, this means that c_{l+j} is determined by the choice of c_l. Since there are $k-1$ formal power series solutions, and we have $k-1$ analytic solutions, we find that each formal solution actually converges in D.

This completes the proof of existence and uniqueness if we are given an integer l such that lQ_f is a subset of the integers. We now turn to the necessity of the multiplicity condition.

IV. As we saw in III, existence of a solution of $h\circ f=\lambda h^k$ is equivalent to the existence of a solution of $\sigma\circ f=\sigma^k$. Suppose σ is single-valued, non-constant, and analytic in D, and satisfies $\sigma \circ f = \sigma^k$ with $\sigma(z) = b_l z^l + \cdots$ where $b_l \neq 0$. We will show lQ_f is a subset of the integers.

We have $\sigma(0) = \sigma(f(0)) = \sigma(0)^k$ and $\sigma(z)^{k^n} = \sigma(f_n(z)) \to \sigma(0)$ for all z in D. Since σ is non-constant, this means that $\sigma(0) = 0$ and $\sigma(D) \subset D$. Choose $\varepsilon' > 0$ so that $|z| < \varepsilon'$ and $\sigma(z) = 0$ implies $z = 0$. Given z_0 in D such that $f_n(z_0) = 0$, choose $\delta > 0$ such that $0 < |z - z_0| < 2\delta$ implies $f_n(z) \neq 0$ and, if $\gamma(t) = z_0 + \delta e^{2\pi i t}$, $0 \leq t \leq 1$, then $|f_n(\gamma(t))| < \varepsilon'$. Now as γ winds around z_0 once, $\sigma(f_n(\gamma(t)))$ winds around zero $lm(z_0, n)$ times, and $\sigma(\gamma(t))$ winds around zero j times where $\sigma(z) = c_j(z - z_0)^j + \cdots$. Since $\sigma(f_n(z)) = \sigma(z)^{k^n}$, we obtain $lm(z_0, n) = jk^n$ so $k^{-n} lm(z_0, n) = j$ an integer. Since $Q_f = \{0\} \cup \{k^{-n} m(z, n)\colon f_n(z) = 0,\ n = 1, 2, \ldots\}$, we see lQ_f is a subset of the integers. □

COROLLARY 1. *If l_0 is the least integer such that $l_0 Q_f$ is a subset of the integers, then every solution of $h \circ f = \lambda h^k$ analytic in the unit disk is of the form $\alpha\sigma^m$ where $\sigma \circ f = \sigma^k$ and $\sigma(z) = b_{l_0} z^{l_0} + \cdots$.*

COROLLARY 2. *If $\{z : f_n(z) = 0$ for some $n\}$ is a finite set then $h \circ f = \lambda h^k$ has non-constant solutions analytic in the unit disk for every $\lambda \neq 0$.*

Proof. Q_f is a finite set of rational numbers. Let l be the least common multiple of the denominators. □

COROLLARY 3. *If $\|f_n\|_\infty < 1$ for some n, then $h \circ f = \lambda h^k$ has non-constant solutions analytic in the unit disk for every $\lambda \neq 0$.*

Proof. Let $\varepsilon > 0$ be small enough that $|z| < \varepsilon$ and $f(z) = 0$ imply $z = 0$. The hypothesis guarantees that, for m_0 large enough, $\|f_{m_0}\| < \varepsilon$ so that, if $m > m_0$ and $f_m(z) = 0$, then $f_{m_0}(z) = 0$. It follows that $k^{m_0} Q_f$ is a subset of the integers. □

We now turn to the special case in which f is real-valued and increasing on $[0, 1)$. (This case has been studied more extensively in the literature, see for example, Szekeres [8].) We pay particular attention to the case in which $\lim_{r \to 1^-} f(r) = 1$, which includes probability generating functions. Theorems 2 and 3 are restatements of results of Kuczma [4] and Ger and Smajdor [3] for the analytic case.

THEOREM 2. *Suppose f is analytic in D, $f(D) \subset D$, $f(z) = a_k z^k + \cdots$, $k \geq 2$, with $a_k > 0$, and $f'(x) \geq 0$ for $0 \leq x < 1$. Then for each $c > 0$ there is a unique function σ, complex analytic near 0, and real analytic on $[0, 1)$ with $\sigma'(0) > 0$, such that $\sigma(f(x)) = c\sigma(x)^k$ for $0 \leq x < 1$. Moreover, $0 < \sigma(x) < 1$ and $\sigma'(x) \geq 0$ for $0 < x < 1$ and, if f satisfies $f'(x) > 0$ for $0 < x < 1$, then σ^{-1} is real analytic (on $(0, \sigma(1-))$) as well.*

Proof. We consider the case $c = 1$ first. Let σ_0 be the function constructed in step I of the proof of Theorem 1. If we choose appropriate branches of the functions in the construction, σ_0 is non-negative on the interval $[0, \varepsilon)$. Since $f'(x) \geq 0$ for $0 \leq x < 1$, we see that $f_n(x) = 0$ if and only if $x = 0$, so we define σ on $[0, 1)$ by $\sigma(x) = [\sigma_0(f_n(x))]^{1/k^n}$, where n is large enough, that $f_n(x) < \varepsilon$, and we take the branch of ω^{1/k^n} that is non-negative on $[0, 1)$. It is easily checked that σ is well-defined, $\sigma(f(x)) = \sigma(x)^k$, and σ has the analyticity properties asserted.

From the functional equation we see that

$$\sigma'(f_n(x))f_n'(x) = k^n[\sigma[\sigma(x)]^{k^n-1}\sigma'(x),$$

so that

$$\sigma'(x) = \frac{\sigma'(f_n(x))f_n'(x)}{k^n[\sigma(x)]^{k^n-1}}.$$

Thus, if $f'(x) > 0$ for $0 < x < 1$, then $f_n'(x) > 0$ for all n and, since $\sigma'(f_n(x)) \neq 0$ for n large enough, we have $\sigma'(x) > 0$ for all x in $(0, 1)$. This means σ^{-1} is well-defined and real analytic on $(0, \sigma(1-))$. An easy computation shows $h(x) = c^{(1-k)^{-1}}\sigma(x)$ satisfies the equation $h \circ f = ch^k$.

As in Theorem 1, the uniqueness is a consequence of the uniqueness of a formal power series solution with $\sigma'(0) > 0$. □

As would be expected, the solution of this functional equation leads to solutions of the classical functional equations on $(0, 1)$. The function $A(x) = (\log k)^{-1}\log|\log \sigma(x)|$ is a real analytic solution of Abel's equation $A \circ f = A + 1$ for $0 < x < 1$. If $\alpha > 0$, $\alpha \neq 1$, $S(x) = [\log \sigma(x)]^P$ where $P = (\log \alpha)(\log k)^{-1}$ is a real analytic solution of Schroeder's equation $S \circ f = \alpha S$ for $0 < x < 1$.

We now turn our attention to fractional iteration.

THEOREM 3. *Suppose f is analytic in D, $f(D) \subset D$, $f(z) = a_k z^k + \cdots$ $k \geq 2$ with $a_k > 0$ and $f'(x) > 0$ for $0 < x < 1$. Then there is a function $F(x, t)$ defined for $0 \leq x < 1$ and $0 \leq t < \infty$, real analytic in each variable such that $F(x, 1) = f(x)$ and $F(F(x, s), t) = F(x, s + t)$ for $0 \leq x < 1$ and $0 \leq s, t < \infty$. Moreover, if $\lim_{x \to 1^-} f(x) = 1$, then F is defined (and has the same properties) for $-\infty < t < \infty$.*

Proof. From Theorem 2 we have that the solution of $\sigma \circ f = \sigma^k$ with $\sigma'(0) > 0$ is real analytic on $(0, 1)$ and has real analytic inverse on $(0, \sigma(1-))$. We define F by $F(x, t) = \sigma^{-1}([\sigma(x)]^{k^t})$. This function is easily seen to have appropriate properties for $t \geqq 0$.

Now, since σ is increasing and $\sigma(x)<1$ for $0<x<1$, $\lim_{x\to 1^-}\sigma(x)$ exists. If $\lim_{x\to 1^-}f(x)=1$, we see from $\sigma(f(x))=\sigma(x)^k$ that $\sigma(1-)=\sigma(1-)^k$ so $\sigma(1-)=1$. It follows that $\sigma^{-1}([\sigma(x)]^{k^t})$ is well defined for $t<0$ as well. $\square$

This theorem says that the discrete semigroup of iterates of f (on $[0,1]$) can be embedded in a continuous semigroup and, if $\lim_{x\to 1^-}f(x)=1$, it can be embedded in a continuous group. In particular this is true if f is a probability generating function ($a_j\geq 0$ and $\sum_{j=k}^{\infty}a_j=1$). It is not difficult to see that this family of iterates is *regular* in the sense of Szekeres [8, p. 216].

REFERENCES

[1] COWEN, C. C., *Iteration and the solution of functional equations for functions analytic in the unit disk.* Trans. Amer. Math. Soc. *265* (1981), 69–95.

[2] KOENIGS, G., *Recherches sur les integrales des certaines équations fonctionelles.* Ann. Ec. Norm. Sup. (3) *1* (1884), supplément, 3–41.

[3] GER, J. and SMAJDOR, A., *Regular iteration in the case of multiplier zero.* Aequationes Math. *7* (1972), 127–131.

[4] KUCZMA, M., *Sur l'équation fonctionnelle de Bottcher.* Mathematica (Cluj) *8* (1966), 279–285.

[5] KUCZMA, M., *Une remarque sur les solutions analytiques d'une équation fonctionnelle.* Colloq. Math. *16* (1967), 93–99.

[6] KUCZMA, M., *Functional equations in a single variable.* Polish Scientific Publishers, Warszawa, 1968.

[7] POMMERENKE, CH., *On the iteration of analytic functions in a half plane.* I. J. London Math. Soc. (2) *19* (1979), 439–447.

[8] SZEKERES, G., *Regular iteration of real and complex functions.* Acta Math. *100* (1958), 203–258.

[9] VALIRON, G., *Fonctions analytiques.* Presses Universitaires de France, Paris, 1954.

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