Commutants of Finite Blaschke Product Multiplication Operators on Bergman Spaces

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joint work with Rebecca Wahl (Butler University)

In this talk $\mathcal{H}$ will denote a Hilbert space of analytic functions on $\mathbb{D}$,

Usual spaces: f analytic in $\mathbb{D}$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$

$$\text{Hardy:} \quad H^2(\mathbb{D}) = H^2 = \{f : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

$$\text{Bergman:} \quad A^2(\mathbb{D}) = A^2 = \{f : \|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{\pi} < \infty\}$$

$$\text{weighted Bergman } (\gamma > -1)\text{:} \quad A^2_\gamma = \{f : \|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\gamma \frac{dA(z)}{\pi} < \infty\}$$

$$\text{weighted Hardy } (\|z^n\| = \omega_n > 0)\text{:} \quad H^2(\omega) = \{f : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \omega_n^2 < \infty\}$$

In these spaces, for α in $\mathbb{D}$, the linear functionals $f \mapsto f(\alpha)$ are bounded.

Being Hilbert spaces, the linear functionals are given by the inner product:

the *reproducing kernel function for* $\mathcal{H}$ is K_α in $\mathcal{H}$ with

$$\langle f, K_\alpha \rangle = f(\alpha) \quad \text{for all} \quad f \in \mathcal{H}$$

For H^2, we have $K_\alpha(z) = (1 - \overline{\alpha}z)^{-1}$

For A^2, we have $K_\alpha(z) = (1 - \overline{\alpha}z)^{-2}$

In this talk, we will consider spaces H^2_κ for $\kappa \geq 1$ which are the weighted Hardy spaces with

$$K_\alpha(z) = (1 - \overline{\alpha}z)^{-\kappa}$$

The spaces H^2_κ include the usual Hardy and Bergman spaces and all the weighted Bergman spaces ($\gamma = \kappa + 2$).

Conversation with Axler made it clear that right generality is to consider Hilbert spaces, $\mathcal{H}$, of functions analytic on $\mathbb{D}$ that satisfy:

(I) The constant function $1(z) \equiv 1$ for z in $\mathbb{D}$ is in $\mathcal{H}$ and $\|1\| = 1$

(II) For α in $\mathbb{D}$, the linear functional $f \mapsto f(\alpha)$ is continuous on $\mathcal{H}$

(III) For ψ in H^∞, operator T_ψ given by $(T_\psi f)(z) = \psi(z) f(z)$ is in $\mathcal{B}(\mathcal{H})$.

(IV) For α in $\mathbb{D}$ and f in $\mathcal{H}$ with $f(\alpha) = 0$, then $f/(z - \alpha)$ is also in $\mathcal{H}$.

- Conditions (I) & (III) say $\mathcal{H}$ and its multiplier algebra contain H^∞
- Condition (II) says $\mathcal{H}$ has kernel functions & its multiplier algebra is H^∞
- For ψ in H^∞, the operator T_ψ in condition (III) is called an *analytic multiplication operator* or an *analytic Toeplitz operator* and conditions imply $\|T_\psi\| = \|\psi\|_\infty$ and this means $\|\psi\| \leq \|\psi\|_\infty$

The Hardy space H^2, the Bergman space A^2, and the standard weight Bergman spaces H^2_κ satisfy Conditions (I), (II), (III), and (IV).

The usual Dirichlet space, and many weighted Dirichlet spaces, do not satisfy all the conditions: not all H^∞ functions are in Dirichlet space!

The Hardy space H^2, the Bergman space A^2, and the standard weight Bergman spaces H^2_κ satisfy Conditions (I), (II), (III), and (IV).

Consequence: if f is in $\mathcal{H}$, ψ is bounded analytic function, and α is in $\mathbb{D}$,

$$\langle f, T_\psi^* K_\alpha \rangle = \langle T_\psi f, K_\alpha \rangle = \psi(\alpha) f(\alpha) = \psi(\alpha)\langle f, K_\alpha \rangle = \langle f, \overline{\psi(\alpha)} K_\alpha \rangle$$

Since f is arbitrary, this means $\quad T_\psi^* K_\alpha = \overline{\psi(\alpha)} K_\alpha$

and every kernel function is an eigenvector for T_ψ^*.

The spectrum of T_ψ is the closure of $\psi(\mathbb{D})$, there no eigenvalues for T_ψ, but the complex conjugate of $\psi(\mathbb{D})$ consists of eigenvalues of T_ψ^*.

Definition:

An *inner function* is a bounded analytic function, ψ, on $\mathbb{D}$ such that

$$\lim_{r \to 1^-} |\psi(re^{i\theta})| = 1 \quad \text{a. e.} \quad d\theta$$

Definition:

A function B is a *Blaschke product of order* n if it can be written as

$$B(z) = \mu \left(\frac{\zeta_1 - z}{1 - \overline{\zeta_1} z} \right) \left(\frac{\zeta_2 - z}{1 - \overline{\zeta_2} z} \right) \cdots \left(\frac{\zeta_n - z}{1 - \overline{\zeta_n} z} \right)$$

where $|\mu| = 1$ and $\zeta_1, \zeta_2, \cdots, \zeta_n$ are points of $\mathbb{D}$.

Blaschke products of order n are inner functions

and map the closed disk n-to-1 onto itself.

For ψ, a non-constant inner function, the multiplication operator T_ψ is a pure isometry on H^2 but is *not* isometric on the Bergman spaces.

Beurling's Theorem (1949):

Let T_z be the operator of multiplication by z on $H^2(\mathbb{D})$. A closed subspace M of $H^2(\mathbb{D})$ is invariant for T_z if and only if there is an inner function ψ such that $M = \psi H^2(\mathbb{D})$.

This result is indicative of the interest in the operator T_z of multiplication by z on $H^2(\mathbb{D})$ and in analytic Toeplitz operators T_ψ on Hilbert spaces of analytic functions more generally.

Definition:

If A is a bounded operator on a space $\mathcal{H}$, the *commutant of* A is the set

$\{A\}' = \{S \in \mathcal{B}(\mathcal{H}) : AS = SA\}$

We have seen for T_z on H^2,

$$\{T_z\}' = \{T_\psi : \psi \in H^\infty\}$$

By the 1970's, there was interest in the more general question,

For ψ in H^∞ and T_ψ an operator on H^2, what is $\{T_\psi\}'$?

or more specifically,

For B a finite Blaschke product and T_B operating on H^2, what is $\{T_B\}'$?

Deddens & Wong's 1973 paper used the fact that, for B a finite Blaschke product, the operator T_B acting on H^2 is a pure isometry to use matrices to characterize operators that commute with T_B.

Shortly thereafter, Thomson's papers and Cowen's papers computed $\{T_B\}'$ from a different perspective:

Fundamental Lemma:

For S a bounded operator on H^2 and ψ in H^∞, these • are equivalent

- *S commutes with T_ψ*
- *For all α in $\mathbb{D}$, $S^* K_\alpha \perp (\psi - \psi(\alpha))H^2$*

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Proof: (Main calculation)

For α in $\mathbb{D}$, ψ in H^∞, and $ST_\psi = T_\psi S$, if f is in H^2,

$$\langle(\psi-\psi(\alpha))f, S^* K_\alpha\rangle = \langle ST_\psi f, K_\alpha\rangle - \psi(\alpha)\langle Sf, K_\alpha\rangle$$

$$= \langle T_\psi Sf, K_\alpha\rangle - \psi(\alpha)\langle Sf, K_\alpha\rangle = \langle \psi Sf, K_\alpha\rangle - \psi(\alpha)\langle Sf, K_\alpha\rangle$$

$$= \psi(\alpha)(Sf)(\alpha) - \psi(\alpha)(Sf)(\alpha) = 0$$

The main results of these papers were to identify some special classes of bounded analytic functions whose Toeplitz operators have commutants that exemplify the possible commutants of analytic Toeplitz operators.

That is, maybe there is a small set $\mathcal{S}$ of H^∞ functions so that for each ψ in H^∞, there is φ in $\mathcal{S}$ so that $\{T_\psi\}' = \{T_\varphi\}'$.

It became clear that, inner functions and covering maps should be part of any such set $\mathcal{S}$ because Toeplitz operators associated with many other H^∞ functions have commutants the same as inner function or covering map Toeplitz operators.

The main results of these papers were to identify some special classes of bounded analytic functions whose Toeplitz operators have commutants that exemplify the possible commutants of analytic Toeplitz operators.

For example, the Fundamental Lemma, immediately implies

If φ and ψ are in H^∞ and there is an analytic function g so that $\varphi = g \circ \psi$, then $\{T_\varphi\}' \supset \{T_\psi\}'$.

So a natural question is: "If $\varphi = g \circ \psi$, when does $\{T_\varphi\}' = \{T_\psi\}'$?"

The main results of these papers were to identify some special classes of bounded analytic functions whose Toeplitz operators have commutants that exemplify the possible commutants of analytic Toeplitz operators.

Theorem: [C., 1978]

If ψ *is a bounded analytic function on the disk* $\mathbb{D}$

and α_0 *is a point of the disk so that the inner factor of* $\psi - \psi(\alpha_0)$

is a finite Blaschke product,

then there is a finite Blaschke product B *so that*

$$\{T_\psi\}' = \{T_B\}'$$

In fact, the Blaschke product B is the "largest" inner function for which there is bounded function g so that $\psi = g \circ B$.

For B a finite Blaschke product of order n, except for $n(n-1)$ points of the disk for which $B(\alpha) = B(\beta)$ and $B'(\beta) = 0$,

$$\left((B - B(\alpha))\, H^2\right)^{\perp} = \text{ span } \{K_{\beta_1}, K_{\beta_2}, \cdots, K_{\beta_n}\}$$

where the points $\alpha = \beta_1$, β_2, $\cdots$, β_n are the n distinct points of $\mathbb{D}$ for which $B(\beta_j) = B(\alpha)$.

The important fact behind this work is that the kernel functions K_α, $K_\alpha(z) = (1 - \overline{\alpha} z)^{-1}$ in H^2 and $K_\alpha(z) = (1 - \overline{\alpha} z)^{-2}$ in A^2, depend conjugate analytically on α, so if A is a linear operator so that AK_α is always in $\left((B - B(\alpha))\, H^2\right)^{\perp}$, then

$$AK_\alpha = \sum_j c_j K_{\beta_j}$$

where the c_j's and the K_{β_j}'s are conjugate analytic in α

For B a finite Blaschke product of order n, except for $n(n-1)$ points of the disk for which $B(\alpha) = B(\beta)$ and $B'(\beta) = 0$,

$$\left(((B - B(\alpha))\, H^2\right)^{\perp} = \text{ span } \{K_{\beta_1}, K_{\beta_2}, \cdots, K_{\beta_n}\}$$

where the points $\alpha = \beta_1$, β_2, $\cdots$, β_n are the n distinct points of $\mathbb{D}$ for which $B(\beta_j) = B(\alpha)$.

Observation:

For the study of commutants of Toeplitz operators, it is more important that a Blaschke product B is an n-to-1 map of $\mathbb{D}$ onto itself than the fact that T_B is a pure isometry on H^2.

Of course, since the points $\alpha = \beta_1, \beta_2, \cdots, \beta_n$ depend on α, we may write them as $\alpha = \beta_1(\alpha), \beta_2(\alpha), \cdots, \beta_n(\alpha)$.

In fact (!), if B is a finite Blaschke product of order n and α is a point of the disk that is *NOT* one of the $n(n-1)$ points of the disk for which

$B(\alpha) = B(\beta)$ and $B'(\beta) = 0$,

the maps $\alpha \mapsto \beta_j(\alpha)$ *are just the* n *branches of the analytic function* $B^{-1} \circ B$ *that is defined and arbitrarily continuable on the disk with the* $n(n-1)$ *exceptional points removed.*

Recall the

Fundamental Lemma:

For S a bounded operator on H^2 and ψ in H^∞, these $\bullet$ are equivalent

- *S commutes with T_ψ*
- *For all α in $\mathbb{D}$, $S^*K_\alpha \perp (\psi - \psi(\alpha))H^2$*

Use ideas about, W, the Riemann surface for $B^{-1} \circ B$ to rewrite this as:

Fundamental Lemma(2):

Let B be a finite Blaschke product. Let F be the set

$F = \{\alpha \in \mathbb{D} : B(\alpha) = B(\beta) \text{ for some } \beta \text{ with } B'(\beta) = 0\}.$

If S is a bounded operator on H^2, then S is in $\{T_B\}'$ if and only if

$S^*K_\alpha = \sum_{j=1}^{n} c_j(\alpha) K_{\beta_j(\alpha)}$ *for each α in $\mathbb{D} \setminus F$.*

We use this to write Sf as a function of α in the disk.

Theorem: (Cowen, 1978). *Let B, F, and W be as above.*

If S is a bounded operator on H^2 that commutes with T_B, then there is a bounded analytic function G on the Riemann surface W so that for f in H^2,

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta, \alpha))\beta'(\alpha)f(\beta(\alpha)) \tag{1}$$

where the sum is taken over the n branches of $B^{-1} \circ B$ at α. Moreover, if α_0 is a zero of order m of B', and ψ_1, ψ_2, $\cdots$, ψ_n is a basis for $\left((B - B(\alpha_0))\, H^2\right)^{\perp}$, then G has the property that

$$\sum G((\beta, \alpha))\beta'(\alpha)\psi_j(\beta(\alpha)) \text{ has a zero of order } m \text{ at } \alpha_0 \tag{2}$$

for $j = 1, 2, \cdots, n$.

Conversely, if G is a bounded analytic function on W that has properties (2) at each zero of B', then (1) defines a bounded linear operator on H^2 with S in $\{T_B\}'$.

Theorem: (C. & Wahl, 2012). *Let B, F, and W be as above.*

If S is a bounded operator on A^2 that commutes with T_B, then there is a bounded analytic function G on the Riemann surface W so that for f in A^2,

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta, \alpha))\beta'(\alpha) f(\beta(\alpha)) \tag{3}$$

where the sum is taken over the n branches of $B^{-1} \circ B$ at α. Moreover, if α_0 is a zero of order m of B', and ψ_1, ψ_2, $\cdots$, ψ_n is a basis for $\left((B - B(\alpha_0))\, A^2\right)^{\perp}$, then G has the property that

$$\sum G((\beta, \alpha))\beta'(\alpha)\psi_j(\beta(\alpha)) \text{ has a zero of order } m \text{ at } \alpha_0 \tag{4}$$

for $j = 1, 2, \cdots, n$.

Conversely, if G is a bounded analytic function on W that has properties (4) at each zero of B', then (3) defines a bounded linear operator on A^2 with S in $\{T_B\}'$.

Theorem: (Cowen, 1978).

If B is a finite Blaschke product and S is a bounded operator on H^2

such that $ST_B = T_BS$,

then for all f in H^∞, Sf is also in H^∞.

Theorem: (C. & Wahl, 2012).

If B is a finite Blaschke product and S is a bounded operator on A^2

such that $ST_B = T_BS$,

then for all f in H^∞, Sf is also in H^∞.

Theorem: (C. & Wahl, 2012).

If B is a finite Blaschke product and S is a bounded operator on A^2

such that $ST_B = T_B S$,

then for all f in H^∞, Sf is also in H^∞.

Corollary:

The commutants of T_B as an operator on H^2 and of T_B as an operator on A^2 are 'the same'.

The bounded analytic functions on the disk are dense in both H^2 and A^2. Since these functions are mapped in the same way as vectors in H^2 and A^2, the operators agree on all vectors common to H^2 and A^2.

In other words, every operator commuting with T_B on the Bergman space is the extension of an operator commuting with T_B on the Hardy space.

Theorem: (C. & Wahl, 2012).

If B is a finite Blaschke product and S is a bounded operator on A^2

such that $ST_B = T_B S$,

then for all f in H^∞, Sf is also in H^∞.

Corollary:

The commutants of T_B as an operator on H^2 and of T_B as an operator on A^2 are 'the same'.

Corollary:

If ψ is a bounded analytic function on the disk $\mathbb{D}$

and α_0 is a point of the disk so that the inner factor of $\psi - \psi(\alpha_0)$

is a finite Blaschke product, there is finite Blaschke product B with

$$\{T_\psi\}' = \{T_B\}' \quad \text{as operators on } A^2$$

Theorem: (C. & Wahl, 2012).

If B is a finite Blaschke product and S is a bounded operator on A^2 such that $ST_B = T_B S$, then for all f in H^∞, Sf is also in H^∞.

Corollary:

The commutants of T_B as an operator on H^2 and of T_B as an operator on A^2 are 'the same'.

Corollary:

If P is a bounded operator acting on H^2 such that $P^2 = P$ and $T_B P = P T_B$, then P is a bounded an operator acting on A^2 such that $P^2 = P$ and $T_B P = P T_B$.

The result

Corollary:

If P is a bounded operator acting on H^2 such that $P^2 = P$ and $T_B P = P T_B$, then P is a bounded an operator acting on A^2 such that $P^2 = P$ and $T_B P = P T_B$.

leads to some obvious, but still unsolved problems: "Which of the projections that commute with T_B on the Bergman space are self-adjoint?"

It is easy to see that many more self-adjoint projections commute with T_B on H^2 than on A^2 because multiplication by B is an isometry in H^2, but not on A^2.

The question "What is $\{T_B, T_B^*\}'$?" is largely unstudied!

Thank You!

Slides available: `http://www.math.iupui.edu/~ccowen`

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