IRRATIONALITY OF $\zeta(3)$

BOGDAN NICA

Comme quelqu'un pourrait dire de moi, que j'ai seulement fait ici un amas de fleurs étrangères: n'y ayant fourni du mien, que le filet à les lier. (Michel de Montaigne, *Essais*)

There is a funerary plaque in Père Lachaise, the largest cemetery in Paris, that carries the engraving

$$1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\cdots\neq\frac{p}{q}.$$

A mathematician would likely recast the engraving in the symbolic form

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\notin\mathbb{Q}.$$

The plaque also carries the name of Roger Apéry (1916–1994).

This is an essay on Apéry's astounding discovery.

Contents

1. Apéritif

The story of the series

$$\sum_{n=1}^{\infty}\frac{1}{n^k}=:\zeta(k)$$

begins in the xiv-th century, with Nicolas Oresme's remarkable discovery that $\zeta(1)=\infty$. In the xviii-th century, Leonhard Euler showed that $\zeta(2)=\pi^2/6$,

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an even more remarkable fact. Subsequently, he calculated $\zeta(4) = \pi^4/90, \zeta(6) = \pi^6/945, \dots$, eventually obtaining the formula

$$\zeta(2k) = \frac{(-1)^{k-1}B_{2k}}{2(2k)!}(2\pi)^{2k}$$

for each positive integer k. (We will discuss Euler's formula, as well as the Bernoulli sequence B_k featuring therein, in Section 5.) The above formula was a great achievement, but also a source of frustration for Euler. For it begged the question of evaluating $\zeta(3)$–more generally, the value of $\zeta(k)$ when k is an odd integer–, a question that Euler repeatedly tried, and failed, to solve. However, Euler made a key conceptual step in viewing $\zeta(k)$ as a function, and studying it not only for integral $k > 1$ but for real $k > 1$; in fact, quite strikingly, he also studied it for real $k < 0$. Another important contribution of Euler was his insight that the zeta function is related to the study of primes. In the xix-th century, the zeta function came into full bloom. Bernhard Riemann extended the meaning of $\zeta(k)$ to complex arguments k, thereby turning ζ into an analytic function throughout the complex plane, except for a simple pole at 1–Oresme's singularity. At this point in history we are catching up with a notational anachronism in our account: the ζ notation was actually introduced by Riemann. The high point of Riemann's contribution is his hypothesis concerning the location of the non-trivial zeroes of the zeta function.

Let us return from Riemann's modern viewpoint to Euler's classical inquiry–that of understanding the values $\zeta(k)$ for integral $k > 1$. Euler's formula implies, in particular, that $\zeta(2k)$ is a rational multiple of π^{2k}. Still in the xix-th century, some three decades after Riemann, Carl Louis Ferdinand von Lindemann showed that π is transcendental. Therefore $\zeta(2k)$ is transcendental.

What is, then, the arithmetic nature of ζ-values at odd integers? rational or irrational? algebraic or transcendental? Besides numerous advances on the Riemann Hypothesis front, the xx-th century brought a first answer to these questions. In 1978, Roger Apéry [6] gave a miraculous proof of the fact that

$$\zeta(3) \notin \mathbb{Q}.$$

Is $\zeta(3)$ transcendental? Is $\zeta(3)$ is a rational multiple of π^3? Are the other ζ-values at odd integers, namely $\zeta(5), \zeta(7), \zeta(9), \dots$, irrational as well? Unknown, unknown, unknown.

Herein, we follow an elegant proof of Apéry's theorem due to Frits Beukers [8]. An interesting and little-known fact is that essentially the same proof as that of Beukers was obtained, independently and concurrently, by Antonio Córdoba (see [11]). Just like Apéry's original proof (see [17] for a lively exposition), Beukers' proof deals with both $\zeta(2)$ and $\zeta(3)$. The proof that $\zeta(2)$ is irrational (Section 3) is interesting here not for the result in itself, but rather for warming-up the ground for the more involved proof that $\zeta(3)$ is irrational (Section 4). In Section 2 we lay out the necessary ingredients.

2. Ingredients

2.1. **The strategy.** The most familiar proof of irrationality, that of $\sqrt{2}$, is an arithmetic proof. The approach to irrationality taken herein has an analytical flavor. It works as follows.

> A real number ξ is irrational provided that there is a non-vanishing sequence $a_n + b_n\xi$, where a_n and b_n are integers, which converges to 0.

Indeed, if ξ were rational then a non-zero integral combination $a_n + b_n\xi$ would be bounded away from zero independently of n. In fact the converse holds as well: if ξ is irrational then there is a non-vanishing sequence $a_n + b_n\xi$, with $a_n, b_n \in \mathbb{Z}$, which converges to 0.

We illustrate this strategy by giving a quick proof of the fact that e is irrational. Let n be a positive integer and multiply the familiar series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!} + \dots$$

by $n!$ to get $n!e = a_n + \theta_n$, where a_n is an integer and

$$\theta_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots.$$

Note that

$$\theta_n < \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots = \frac{1}{n}.$$

Thus, the non-vanishing sequence $a_n - n!e = \theta_n$ converges to 0. It follows that e is irrational.

2.2. **Legendre polynomials.** Key to the arguments that follow are the Legendre polynomials[1] $P_n(x)$, defined by

$$P_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}\big(x^n(1-x)^n\big).$$

While this implicit definition–known as the Rodrigues formula–is the most convenient for our purposes, an explicit formula for $P_n(x)$ is easy to write down. Using the binomial formula on $x^n(1-x)^n$, and then differentiating n times, we find

$$P_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k} x^k.$$

In particular, we see that $P_n(x)$ has integral coefficients. The first few Legendre polynomials are $P_0(x) = 1$, $P_1(x) = 1 - 2x$, $P_2(x) = 1 - 6x + 6x^2$, $P_3(x) = 1 - 12x + 30x^2 - 20x^3$.

[1] Strictly speaking, these are shifted Legendre polynomials. The standard Legendre polynomials are tailored to the symmetric interval $[-1, 1]$, whereas here we work over $[0, 1]$.

The Legendre polynomials are friendly to integration by parts over the interval $[0,1]$. For $j = 0, 1, \dots, n-1$, the j-th derivative of $x^n(1-x)^n$ vanishes at the endpoints 0 and 1. So for a smooth function g we can write:

$$\begin{aligned}\int_0^1 P_n(x)g(x)\,dx &= \int_0^1 \frac{1}{n!}\frac{d^n}{dx^n}\big(x^n(1-x)^n\big)\,g(x)\,dx\\ &= -\int_0^1 \frac{1}{n!}\frac{d^{n-1}}{dx^{n-1}}\big(x^n(1-x)^n\big)\,g'(x)\,dx\end{aligned}$$

and, upon repeating this derivative shifting n times and taking the absolute value, we obtain:

$$\left|\int_0^1 P_n(x)g(x)\,dx\right| = \left|\int_0^1 \frac{x^n(1-x)^n}{n!}\,g^{(n)}(x)\,dx\right|. \tag{1}$$

Although we will not use this fact in what follows, it is worth pointing out that (1) implies, in particular, that the Legendre polynomials are orthogonal:

$$\int_0^1 P_n(x)P_m(x)\,dx = 0$$

whenever $m \neq n$. Indeed, we may assume without loss of generality that $m < n$. We apply (1) to $g(x) = P_m(x)$; as $P_m(x)$ is a polynomial of degree m, its n-th derivative vanishes and then so does the right-hand side of (1).

2.3. **The least common multiple of** $1, 2, \dots, n$**.** Consider the integral sequence

$$d_n = \operatorname{lcm}(1, 2, \dots, n).$$

We will need the following bound: for n large enough,

$$d_n < 3^n. \tag{2}$$

Indeed, notice that d_n is the product of prime powers p^{α_p}, over all primes $p \leq n$, where α_p is the greatest integer with $p^{\alpha_p} \leq n$. So $d_n \leq n^{\pi(n)}$, where $\pi(n)$ denotes the number of primes no greater than n. The Prime Number Theorem says that

$$1 = \lim_{n\to\infty} \frac{\pi(n)\ln n}{n} = \lim_{n\to\infty} \frac{\ln n^{\pi(n)}}{n}.$$

Whence, for n sufficiently large, we have $\ln n^{\pi(n)} < n \ln 3$, that is $n^{\pi(n)} < 3^n$; the bound (2) follows.

Hanson [12] has given an elementary proof that the bound (2) holds for all n. His argument is, however, not particularly elegant. The appeal to the Prime Number Theorem seems more conceptual.

3. Irrationality of $\zeta(2)$

3.1. **An integral formula.** As a way to introduce a viewpoint and a few shorthands, used throughout, we start with the following formula:

$$\zeta(2) = \iint_{\square} \frac{dxdy}{1-xy}. \tag{3}$$

The first shorthand is that, throughout this write-up, $\iint_\square$ denotes the improper double integral over the unit square $[0,1]\times[0,1]$. To prove (3), we use the expansion

$$\frac{1}{1-xy} = \sum_{n=0}^{\infty} x^n y^n$$

and then we integrate:

$$\iint_\square \frac{dxdy}{1-xy} = \sum_{n=0}^{\infty} \iint_\square x^n y^n \, dxdy = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \zeta(2).$$

Along the way, we have interchanged the integral and the infinite sum, and this can be justified by the monotone convergence theorem. The second shorthand is that we will frequently skim over this technicality, as well as others: interchanging the integral and the derivative, and freely using improper integrals.

In what follows, we will encounter several variations on (3) and its proof.

3.2. **An elementary evaluation.** Following Córdoba [11], we evaluate $\zeta(2)$ by using the following variation on the integral formula (3):

$$3\,\zeta(2) = \int_{-1}^{1}\int_{-1}^{1} \frac{dxdy}{1-x^2y^2}. \tag{4}$$

Indeed, we can adapt the argument for (3) by writing

$$\int_{-1}^{1}\int_{-1}^{1} \frac{dxdy}{1-x^2y^2} = \sum_{n=0}^{\infty} \int_{-1}^{1}\int_{-1}^{1} x^{2n}y^{2n}\,dxdy = 4\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

But the right-hand side equals $3\,\zeta(2)$, since

$$\zeta(2) = \sum_{n=0}^{\infty}\frac{1}{n^2} = \sum_{n=0}^{\infty}\frac{1}{(2n)^2} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\zeta(2)}{4} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

We now evaluate the double integral in (4) by substituting $x := \tanh(u)$, $y := \tanh(v)$. We calculate

$$\begin{aligned}\frac{dxdy}{1-x^2y^2} &= \frac{dudv}{\cosh^2(u)\cosh^2(v) - \sinh^2(u)\sinh^2(v)} \\ &= \frac{dudv}{\cosh(u+v)\cosh(u-v)}\end{aligned}$$

and so

$$3\,\zeta(2) = \int_{-1}^{1}\int_{-1}^{1}\frac{dxdy}{1-x^2y^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dudv}{\cosh(u+v)\cosh(u-v)}.$$

The next change of variables, $z := u+v$ and $t := u-v$, is self-evident. Since the Jacobian of this linear transformation is 2, that is to say, it contracts area by a factor of 2, we obtain

$$3\,\zeta(2) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dzdt}{\cosh(z)\cosh(t)} = 2\left(\int_{-\infty}^{\infty}\frac{dt}{\cosh t}\right)^2.$$

The latter integral is a very simple exercise in calculus. The substitution $t := \ln w$ gives

$$\int_{-\infty}^{\infty} \frac{dt}{\cosh t} = \int_0^{\infty} \frac{dw}{1+w^2} = \arctan w\Big|_0^{\infty} = \frac{\pi}{2}.$$

We conclude that

$$\zeta(2) = \frac{\pi^2}{6}.$$

Two other arguments based on related integral formulas appear in [3, pp.55-57]. Arguably, the above argument–due, once again, to Antonio Córdoba–is closer to being a proof from The Book!

3.3. **A proof of irrationality.** The evaluation of $\zeta(2)$ and the fact that π is transcendental immediately imply that $\zeta(2)$ is irrational–in fact, transcendental. We now work out a different approach to the irrationality of $\zeta(2)$, one that we can later on adapt for $\zeta(3)$. In the case of $\zeta(3)$, we will no longer have the luxury of an explicit evaluation.

We need two preparatory facts. The first one is a relative of (3) and its proof.

Lemma 3.1. *Let k and j be non-negative integers. Then*

$$\iint_{\square} \frac{x^k y^j}{1-xy}\, dxdy = \begin{cases} \zeta(2) + a/d_k^2 & \text{if } k = j \\ a'/d_k^2 & \text{if } k > j \end{cases}$$

for some integers a, a'.

Proof. We have

$$\iint_{\square} \frac{x^k y^j}{1-xy}\, dxdy = \sum_{n=0}^{\infty} \iint_{\square} x^{n+k} y^{n+j}\, dxdy = \sum_{n=0}^{\infty} \frac{1}{n+k+1} \cdot \frac{1}{n+j+1}.$$

If $k = j$ then

$$\iint_{\square} \frac{x^k y^k}{1-xy}\, dxdy = \sum_{n=0}^{\infty} \frac{1}{(n+k+1)^2} = \zeta(2) - \frac{1}{1^2} - \cdots - \frac{1}{k^2}$$

and the latter term is of the form $\zeta(2) + a/d_k^2$ for some integer a.

If $k > j$ then, by telescoping, we get

$$\begin{aligned}\iint_{\square} \frac{x^k y^j}{1-xy}\, dxdy &= \frac{1}{k-j} \sum_{n=0}^{\infty} \left(\frac{1}{n+j+1} - \frac{1}{n+k+1} \right) \\ &= \frac{1}{k-j} \left(\frac{1}{j+1} + \cdots + \frac{1}{k} \right)\end{aligned}$$

and the latter rational number can be expressed as a'/d_k^2 for some integer a'. $\square$

Lemma 3.2. *For all $0 \le x, y \le 1$ we have*

$$\frac{x(1-x)y(1-y)}{1-xy} < \frac{1}{10}.$$

Proof. Let $f(x,y)$ be the function given in the lemma. Notice first that f vanishes on the boundary of $[0,1]\times[0,1]$. The function f is not defined for $(1,1)$, but we have $f(x,y)\to 0$ as $x,y\nearrow 1$.

To find the maximum of f in the unit square, we solve the system

$$\frac{\partial}{\partial x}f(x,y)=0=\frac{\partial}{\partial y}f(x,y)$$

in $(0,1)\times(0,1)$. The system amounts to

$$1-2x+yx^2=0=1-2y+xy^2.$$

Note first that $1-2x+yx^2=1-2y+xy^2$ implies $x=y$. The equation to solve is then the cubic $x^3-2x+1=0$, whose roots are $1,(-1\pm\sqrt{5})/2$. We conclude that the solution is $x=y=(\sqrt{5}-1)/2$. These are the coordinates of the point where f achieves its maximum value $\left((\sqrt{5}-1)/2\right)^5=0.0901...<0.1$. $\square$

Theorem 3.3. *$\zeta(2)$ is irrational.*

Proof. Let n be a positive integer, and consider the following integral

$$I_n=\iint_\square \frac{P_n(x)(1-y)^n}{1-xy}\,dxdy.$$

On the one hand, as P_n is a polynomial with integer coefficients, we infer from Lemma 3.1 that

$$I_n=\frac{a_n+b_n\zeta(2)}{d_n^2}$$

for some integers a_n,b_n. On the other hand, using (1) we have

$$\begin{aligned}
|I_n| &= \left|\int_0^1 P_n(x)\left(\int_0^1\frac{(1-y)^n}{1-xy}dy\right)dx\right|\\
&= \left|\int_0^1 \frac{x^n(1-x)^n}{n!}\frac{d^n}{dx^n}\left(\int_0^1\frac{(1-y)^n}{1-xy}\,dy\right)dx\right|\\
&= \left|\int_0^1 \frac{x^n(1-x)^n}{n!}\left(\int_0^1\frac{d^n}{dx^n}\left(\frac{(1-y)^n}{1-xy}\right)dy\right)dx\right|\\
&= \left|\int_0^1 \frac{x^n(1-x)^n}{n!}\left(\int_0^1\frac{n!y^n(1-y)^n}{(1-xy)^{n+1}}\,dy\right)dx\right|\\
&= \iint_\square \frac{x^n(1-x)^ny^n(1-y)^n}{(1-xy)^{n+1}}\,dxdy.
\end{aligned}$$

We learn two facts from this computation: firstly, that $I_n\neq 0$; secondly, thanks to Lemma 3.2, that

$$|I_n|<(0.1)^n\iint_\square\frac{dxdy}{1-xy}=(0.1)^n\zeta(2).$$

Combining these facts about I_n, we find that

$$0<|a_n+b_n\zeta(2)|<d_n^2\,(0.1)^n\zeta(2).$$

Now, for n sufficiently large, we have $d_n < 3^n$ and so the upper bound is at most $(0.9)^n\zeta(2)$. Thus, the non-vanishing sequence $a_n + b_n\zeta(2)$ converges to 0, which implies that $\zeta(2)$ is irrational. $\square$

4. Irrationality of $\zeta(3)$

The irrationality of $\zeta(2)$, discussed in detail in the previous section, provides us with the blueprint for attacking $\zeta(3)$. We start by adapting the preliminary lemmas.

Lemma 4.1. *We have*

$$2\,\zeta(3) = \iint_\square \frac{-\ln(xy)}{1-xy}\,dxdy.$$

More generally, if k and j are non-negative integers, then

$$\iint_\square \frac{-x^k y^j \ln(xy)}{1-xy}\,dxdy = \begin{cases} 2\,\zeta(3) + a/d_k^3 & \text{if } k = j \\ a'/d_k^3 & \text{if } k > j \end{cases}$$

for some integers a, a'.

Proof. Let $t \geq 0$. As in the proof of Lemma 3.1, we have

$$\iint_\square \frac{x^{k+t}y^{k+t}}{1-xy}\,dxdy = \sum_{n=0}^{\infty} \frac{1}{(n+k+t+1)^2}.$$

By differentiating with respect to t we obtain

$$\iint_\square \frac{x^{k+t}y^{k+t}\ln(xy)}{1-xy}\,dxdy = -2\sum_{n=0}^{\infty} \frac{1}{(n+k+t+1)^3}$$

which, at $t = 0$, gives

$$\iint_\square \frac{-x^k y^k \ln(xy)}{1-xy}\,dxdy = 2\sum_{n=0}^{\infty} \frac{1}{(n+k+1)^3}.$$

The right-hand side evaluates to $2\,\zeta(3)$ when $k = 0$, thereby yielding the first identity of the lemma. In general, the right-hand side can be written

$$2\sum_{n=0}^{\infty} \frac{1}{(n+k+1)^3} = 2\,\zeta(3) - 2\Big(\frac{1}{1^3} + \cdots + \frac{1}{k^3}\Big) = 2\,\zeta(3) + \frac{a}{d_k^3}$$

for some integer a.

Now let $k > j$ be integers. Picking up again a formula from the proof of Lemma 3.1, we have

$$\iint_\square \frac{x^{k+t}y^{j+t}}{1-xy}\,dxdy = \frac{1}{k-j}\left(\frac{1}{j+t+1} + \cdots + \frac{1}{k+t}\right).$$

Differentiating with respect to t gives

$$\iint_\square \frac{x^{k+t}y^{j+t}\ln(xy)}{1-xy}\,dxdy = \frac{-1}{k-j}\left(\frac{1}{(j+t+1)^2} + \cdots + \frac{1}{(k+t)^2}\right)$$

which, for $t = 0$, reads as follows:

$$\iint_{\square} \frac{-x^k y^j \ln(xy)}{1 - xy}\, dxdy = \frac{1}{k-j}\left(\frac{1}{(j+1)^2} + \cdots + \frac{1}{k^2}\right).$$

The right-hand side is of the form a'/d_k^3 for some integer a'. $\square$

Lemma 4.2. *For all* $0 \le x, y, w \le 1$ *we have*

$$\frac{x(1-x)y(1-y)w(1-w)}{1-(1-xy)w} < \frac{3}{100}.$$

Proof. Let $f(x, y, w)$ denote the given rational function. We note that f vanishes on the boundary of the unit cube $[0, 1] \times [0, 1] \times [0, 1]$. Strictly speaking, on the edges $x = 0, w = 1$ and $y = 0, w = 1$ the function f is not defined, but we have $f(x, y, w) \to 0$ as $x \searrow 0$ and $w \nearrow 1$, or as $y \searrow 0$ and $w \nearrow 1$.

To find the maximum of f over the unit cube, we solve the following system in $(0, 1) \times (0, 1) \times (0, 1)$:

$$\frac{\partial}{\partial x} f(x, y, w) = \frac{\partial}{\partial y} f(x, y, w) = \frac{\partial}{\partial w} f(x, y, w) = 0.$$

Straightforward manipulations lead to

$$\begin{cases} (1-2x) - (1-2x+x^2y)w = 0, \\ (1-2y) - (1-2y+xy^2)w = 0, \\ 1 - 2w + (1-xy)w^2 = 0. \end{cases}$$

The first two equations imply that $x = y$. The system simplifies to

$$\begin{cases} (1-2x) - (1-2x+x^3)w = 0, \\ 1 - 2w + (1-x^2)w^2 = 0. \end{cases}$$

The last equation can be written $(1-w)^2 = (xw)^2$, whence $1 - w = xw$. In combination with the first equation, we quickly reach the quadratic relation $x^2 + 2x - 1 = 0$. Hence $x = y = \sqrt{2} - 1$ and $w = 1/\sqrt{2}$; these are the coordinates of the point where f achieves its maximum value $(\sqrt{2} - 1)^4 = 0.0294... < 0.03$. $\square$

Here it comes! The main result.

Theorem 4.3 (Apéry)**.** $\zeta(3)$ *is irrational.*

Proof. Let n be a positive integer. Consider the integral

$$I_n = \iint_{\square} \frac{-P_n(x)P_n(y)\ln(xy)}{1-xy}\, dxdy.$$

As the Legendre polynomial P_n has integral coefficients, we infer with the help of Lemma 4.1 that

$$I_n = \frac{a_n + b_n\zeta(3)}{d_n^3}$$

for some integers a_n, b_n. On the other hand, by using the integral formula

$$\frac{-\ln(xy)}{1-xy} = \int_0^1 \frac{1}{1-(1-xy)z}\, dz$$

as well as (1), we have

$$\begin{aligned}
|I_n| &= \left| \int_0^1 P_n(x) \left(\iint_\square \frac{P_n(y)}{1-(1-xy)z}\, dydz \right) dx \right| \\
&= \left| \int_0^1 \frac{x^n(1-x)^n}{n!} \frac{d^n}{dx^n} \left(\iint_\square \frac{P_n(y)}{1-(1-xy)z}\, dydz \right) dx \right| \\
&= \left| \int_0^1 \frac{x^n(1-x)^n}{n!} \left(\iint_\square \frac{d^n}{dx^n} \left(\frac{P_n(y)}{1-(1-xy)z} \right) dydz \right) dx \right| \\
&= \left| \int_0^1 \frac{x^n(1-x)^n}{n!} \left(\iint_\square \frac{(-1)^n n! P_n(y) y^n z^n}{\big(1-(1-xy)z\big)^{n+1}}\, dydz \right) dx \right| \\
&= \left| \int_0^1 P_n(y) \left(\iint_\square \frac{x^n(1-x)^n y^n z^n}{\big(1-(1-xy)z\big)^{n+1}}\, dxdz \right) dy \right|.
\end{aligned}$$

At this point we make the change of variables from z to w described by

$$1-w = \frac{xyz}{1-(1-xy)z}.$$

A straightforward but somewhat tedious manipulation–which reveals that the above change of variables is actually an involution–shows that

$$\frac{dz}{1-(1-xy)z} = -\frac{dw}{1-(1-xy)w},$$

and so we can continue as follows:

$$\begin{aligned}
|I_n| &= \left| \int_0^1 P_n(y) \left(\iint_\square \frac{(1-x)^n(1-w)^n}{1-(1-xy)w}\, dxdw \right) dy \right| \\
&= \left| \int_0^1 \frac{y^n(1-y)^n}{n!} \frac{d^n}{dy^n} \left(\iint_\square \frac{(1-x)^n(1-w)^n}{1-(1-xy)w}\, dxdw \right) dy \right| \\
&= \left| \int_0^1 \frac{y^n(1-y)^n}{n!} \left(\iint_\square \frac{d^n}{dy^n} \left(\frac{(1-x)^n(1-w)^n}{1-(1-xy)w} \right) dxdw \right) dy \right| \\
&= \left| \int_0^1 \frac{y^n(1-y)^n}{n!} \left(\iint_\square \frac{(-1)^n n!(1-x)^n(1-w)^n x^n w^n}{\big(1-(1-xy)w\big)^{n+1}}\, dxdw \right) dy \right| \\
&= \iiint_0^1 \frac{x^n(1-x)^n y^n(1-y)^n w^n(1-w)^n}{\big(1-(1-xy)w\big)^{n+1}}\, dxdydw.
\end{aligned}$$

Along the way, we have used (1) once again.

Firstly, we see that $I_n \neq 0$. Secondly, the estimate of Lemma 4.2 gives

$$|I_n| \leq (0.03)^n \iiint_0^1 \frac{dxdydw}{1-(1-xy)w} = (0.03)^n \iint_\square \frac{-\ln(xy)}{1-xy}\,dxdy = (0.03)^n\,2\,\zeta(3).$$

Taken together, these facts about I_n imply that

$$0 < |a_n + b_n\zeta(3)| \leq d_n^3\,(0.03)^n\,2\,\zeta(3).$$

For sufficiently large n, we have $d_n < 3^n$ and so the upper bound is less than $(0.81)^n\,2\,\zeta(3)$. So the non-vanishing sequence $a_n + b_n\zeta(3)$ converges to 0, which implies that $\zeta(3)$ is irrational. □

5. Euler's formula

In this section we prove the following formula, which gives a closed form evaluation of the zeta values at even positive integers.

Theorem 5.1. *Let k be a positive integer. Then*

$$\zeta(2k) = \frac{(-1)^{k-1}B_{2k}}{2(2k)!}(2\pi)^{2k}. \tag{5}$$

Euler's formula involves the Bernoulli numbers B_n, and our account starts with a discussion of this important sequence. Presumably, no simple closed formula for the Bernoulli numbers exists, which means that referring to (5) as a closed formula is somewhat debatable. At any rate, the Bernoulli numbers are rational. So the qualitative interpretation of (5) is that $\zeta(2k)$ is a rational multiple of π^{2k}. This is interesting enough on its own; it implies, for instance, that $\zeta(2k)$ is transcendental.

5.1. Bernoulli numbers.

The Bernoulli sequence B_n is defined by the Taylor expansion

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \tag{6}$$

By reading off the coefficient of x^n in the identity

$$1 = \frac{x}{e^x-1}\cdot\frac{e^x-1}{x} = \left(\sum_{n=0}^{\infty} B_n\frac{x^n}{n!}\right)\cdot\left(\sum_{n=0}^{\infty}\frac{x^n}{(n+1)!}\right)$$

we obtain the recurrence

$$\binom{n+1}{1}B_n + \binom{n+1}{2}B_{n-1} + \cdots + \binom{n+1}{n}B_1 + B_0 = 0, \tag{7}$$

with initial value $B_0 = 1$. An important upshot of the recurrence (7) is that the Bernoulli numbers are rational numbers. The recurrence (7) also allows us to compute the first few terms in the Bernoulli sequence.

B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}
1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$

It appears that $B_n = 0$ for odd $n > 1$, and this is indeed the case: the function

$$\frac{x}{e^x - 1} + \frac{x}{2} = \frac{x}{2} \cdot \frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}}$$

is even, so $B_1 = -1/2$ while $B_3, B_5, B_7, \dots$ vanish.[2]

Historically, Bernoulli numbers first appeared in what we now call the Faulhaber formula:

$$1^n + 2^n + \dots + N^n = \frac{1}{n+1} \sum_{j=0}^{n} (-1)^j \binom{n+1}{j} B_j\, N^{n-j+1}. \tag{8}$$

The right-hand side of (8) is thus a polynomial of degree $n+1$ in N with rational coefficients, of the form

$$\frac{1}{n+1}\, N^{n+1} + \frac{1}{2}\, N^n + \frac{n}{12}\, N^{n-1} + O(N^{n-3}).$$

The formula (8) is named after Johann Faulhaber who, by 1630, had obtained it–in explicit form–up to $n = 17$. In fact, according to Knuth [14], Faulhaber may have worked it up to $n = 23$. The mysterious coefficients that showed up in Faulhaber's results were elucidated much later by Jakob Bernoulli–whence the terminology for the sequence B_n.

Let us prove (8). Put $S_n(N) = 1^n + 2^n + \dots + N^n$. Then

$$\sum_{n=0}^{\infty} S_n(N)\, \frac{x^n}{n!} = \sum_{m=1}^{N} \sum_{n=0}^{\infty} \frac{(mx)^n}{n!} = \sum_{m=1}^{N} e^{mx} = e^x \cdot \frac{e^{Nx} - 1}{e^x - 1}.$$

Next, we write

$$e^x \cdot \frac{e^{Nx} - 1}{e^x - 1} = \frac{e^{Nx} - 1}{x} \cdot \frac{-x}{e^{-x} - 1}$$

and we Taylor expand each factor:

$$\frac{e^{Nx} - 1}{x} = \sum_{n=1}^{\infty} N^n\, \frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty} N^{n+1}\, \frac{x^n}{(n+1)!}$$

respectively

$$\frac{-x}{e^{-x} - 1} = \sum_{n=0}^{\infty} B_n\, \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n B_n\, \frac{x^n}{n!}.$$

[2] In what concerns B_n for even n, an interesting arithmetic information is given by the Clausen–von Staudt theorem: the denominator of B_n is $\prod_{\substack{p-1 \mid n \\ p \text{ prime}}} p$.

Therefore

$$\sum_{n=0}^{\infty} S_n(N)\,\frac{x^n}{n!} = \left(\sum_{n=0}^{\infty} N^{n+1}\,\frac{x^n}{(n+1)!}\right)\cdot\left(\sum_{n=0}^{\infty}(-1)^n B_n\,\frac{x^n}{n!}\right).$$

We conclude by comparing the coefficients of $x^n/n!$: on the left-hand side, the coefficient is $S_n(N)$, whereas on the right-hand side it is

$$n!\sum_{j=0}^{n}\frac{(-1)^j B_j}{j!}\,\frac{N^{n+1-j}}{(n+1-j)!} = \frac{1}{n+1}\sum_{j=0}^{n}(-1)^j\binom{n+1}{j}B_j\,N^{n-j+1}.$$

5.2. **Proof of Euler's formula.** We will derive (5) from two rather similar convolution identities, one for the (weighted) Bernoulli numbers B_{2k}, and the other for the zeta values $\zeta(2k)$.

Lemma 5.2. *Put* $b_{2k} = B_{2k}/(2k)!$. *Then, for all* $k \geq 2$, *we have*

$$-(2k+1)b_{2k} = \sum_{j=1}^{k-1} b_{2j}b_{2k-2j}. \tag{9}$$

Lemma 5.3. *For all* $k \geq 2$, *we have*

$$\left(k+\frac{1}{2}\right)\zeta(2k) = \sum_{j=1}^{k-1}\zeta(2j)\zeta(2k-2j). \tag{10}$$

The upshot is that $-b_{2k}$ and $2\zeta(2k)$ satisfy the same convolution identity, whence

$$\frac{2\zeta(2k)}{-b_{2k}} = \left(\frac{2\zeta(2)}{-b_2}\right)^k$$

for all $k \geq 1$. Since $\zeta(2) = \pi^2/6$ and $b_2 = B_2/2! = 1/12$, we deduce that

$$\zeta(2k) = -\frac{1}{2}\,\frac{B_{2k}}{(2k)!}\,(-4\pi^2)^k = \frac{(-1)^{k-1}}{2}\,\frac{B_{2k}}{(2k)!}\,(2\pi)^{2k},$$

thereby proving Euler's formula (5).

If one is only interested in the fact that $\zeta(2k)$ is a rational multiple of π^{2k}, then one can directly use Lemma 5.3; this way, Bernoulli numbers and Euler's formula can be bypassed. For (9) implies that $\zeta(2k)$ is a rational multiple of $\zeta(2)^k$, and we already know $\zeta(2)$ to be a rational multiple of π^2.

We now turn to the proof of the two lemmas.

Proof of Lemma 5.2. We start by writing (6) in the form:

$$\frac{x}{e^x-1} = -\frac{x}{2} + \sum_{k=0}^{\infty} b_{2k}\,x^{2k}. \tag{11}$$

We will compute in two ways the coefficient of x^{2k}, for $k \geq 2$, in the expansion of $x^2/(e^x-1)^2$. By squaring (11) we see that the desired coefficient is

$$\sum_{j=0}^{k} b_{2j}b_{2k-2j} = 2b_{2k} + \sum_{j=1}^{k-1} b_{2j}b_{2k-2j}.$$

On the other hand, we can differentiate (11) term by term to obtain

$$\frac{1-x}{e^x-1}-\frac{x}{(e^x-1)^2}=-\frac{1}{2}+\sum_{k=1}^{\infty} 2kb_{2k}\, x^{2k-1}.$$

We multiply through by x and we rearrange:

$$\frac{x^2}{(e^x-1)^2}=(1-x)\cdot\frac{x}{e^x-1}+\frac{x}{2}-\sum_{k=1}^{\infty} 2kb_{2k}\, x^{2k}.$$

We read off again the coefficient of x^{2k} for $k\geq 2$ on the right-hand side. Upon using (11), the first term contributes b_{2k}, whereas the last term contributes $-2kb_{2k}$; overall, the desired coefficient is $(1-2k)b_{2k}$.

The two coefficient formulas say that

$$(1-2k)b_{2k}=2b_{2k}+\sum_{j=1}^{k-1} b_{2j}b_{2k-2j}$$

and (9) follows. □

The following proof of (10) is due to Williams [19].

Proof of Lemma 5.3. Let N be a positive integer and consider the finite sum

$$w(N)=\sum_{j=1}^{k-1}\bigg(\sum_{n=1}^{N}\frac{1}{n^{2j}}\bigg)\bigg(\sum_{m=1}^{N}\frac{1}{m^{2k-2j}}\bigg).$$

Clearly, $w(N)$ converges to the right-hand side of (10) as $N\to\infty$. Our aim is to manipulate $w(N)$ so as to show that

$$(*)\qquad w(N)\to\Big(k+\frac{1}{2}\Big)\zeta(2k)\qquad\text{as } N\to\infty.$$

The desired identity (10) will follow.

We rewrite

$$w(N)=\sum_{n,m=1}^{N}\sum_{j=1}^{k-1}\frac{1}{n^{2j}m^{2k-2j}},$$

and we split $w(N)$ into two parts, according to whether $n=m$ or $n\neq m$. The first part is immediate:

$$w(N)\Big|_{m=n}=(k-1)\sum_{n=1}^{N}\frac{1}{n^{2k}}.$$

When $n\neq m$, we have

$$\begin{aligned}\sum_{j=1}^{k-1}\frac{1}{n^{2j}m^{2k-2j}}&=\frac{1}{n^{2k-2}m^{2k-2}}\cdot\frac{n^{2k-2}-m^{2k-2}}{n^2-m^2}\\&=\frac{1}{m^{2k-2}(n^2-m^2)}+\frac{1}{n^{2k-2}(m^2-n^2)}.\end{aligned}$$

Thus, taking advantage of the symmetry, we can write

$$w(N)\Big|_{m\neq n} = \sum_{\substack{n,m=1\\ n\neq m}}^{N} \sum_{j=1}^{k-1} \frac{1}{n^{2j}m^{2k-2j}} = 2\sum_{\substack{n,m=1\\ n\neq m}}^{N} \frac{1}{n^{2k-2}(m^2-n^2)}.$$

For each index n we have

$$\begin{aligned}\sum_{\substack{m=1\\ m\neq n}}^{N} \frac{1}{m^2-n^2} &= \frac{1}{2n}\left(\sum_{\substack{m=1\\ m\neq n}}^{N} \frac{1}{m-n} - \sum_{\substack{m=1\\ m\neq n}}^{N} \frac{1}{m+n}\right)\\ &= \frac{1}{2n}\left(\sum_{m=1}^{N-n}\frac{1}{m} - \sum_{m=1}^{n-1}\frac{1}{m} - \sum_{m=n+1}^{2n-1}\frac{1}{m} - \sum_{m=2n+1}^{n+N}\frac{1}{m}\right)\\ &= \frac{1}{2n}\left(\sum_{m=1}^{N-n}\frac{1}{m} - \sum_{m=1}^{n+N}\frac{1}{m} + \frac{3}{2n}\right) = \frac{3}{4n^2} - s_n\end{aligned}$$

where

$$s_n = \frac{1}{2n}\sum_{m=N-n+1}^{n+N}\frac{1}{m}.$$

Thus

$$w(N)\Big|_{m\neq n} = 2\sum_{n=1}^{N}\frac{1}{n^{2k-2}}\left(\frac{3}{4n^2} - s_n\right) = \frac{3}{2}\sum_{n=1}^{N}\frac{1}{n^{2k}} - 2\sum_{n=1}^{N}\frac{s_n}{n^{2k-2}}.$$

Now putting the two parts together, we find

$$w(N) = w(N)\Big|_{m=n} + w(N)\Big|_{m\neq n} = \left(k+\frac{1}{2}\right)\sum_{n=1}^{N}\frac{1}{n^{2k}} - 2\sum_{n=1}^{N}\frac{s_n}{n^{2k-2}}.$$

We finally see $(*)$ emerging; all that is left to check is that

$$(**) \qquad \sum_{n=1}^{N}\frac{s_n}{n^{2k-2}} \to 0 \qquad \text{as } N\to\infty.$$

We note the rough estimate $s_n < 1/(N-n+1)$, which will suffice for our purposes. Indeed, s_n is the average of $2n$ terms, the largest of which is $1/(N-n+1)$. Then

$$\begin{aligned}\sum_{n=1}^{N}\frac{s_n}{n} &< \sum_{n=1}^{N}\frac{1}{n(N-n+1)} = \frac{1}{N+1}\sum_{n=1}^{N}\left(\frac{1}{n}+\frac{1}{N-n+1}\right)\\ &= \frac{2}{N+1}\sum_{n=1}^{N}\frac{1}{n} < \frac{2}{N+1}(\ln N + 1)\end{aligned}$$

and the latter expression converges to 0 as $N\to\infty$. Since $k\geq 2$, and so $2k-2\geq 2$, $(**)$ comfortably follows. □

Another road to Euler's formula (5)–an approach which is quite close to Euler's original approach–is explained in [3, Ch.26]; see also [17, pp.196–197].

6. Digestifs

In this closing section we give a brief overview of several facets of Apéry's incredible discovery: his original argument and the memorable lecture in which he outlined it; his personality, mathematical philosophy, and standing at the time within French mathematics; subsequent results inspired by Apéry's result.

6.1. **The proof.** Apéry's approach to the irrationality of $\zeta(3)$ is sketched in [6], an announcement which is nothing more than a short list of statements that are lacking explanations. Later expositions, notably those by Cohen [10] and by van der Poorten [17], have sought to spell out missing details and to elucidate the miracle. See also Havil [13, Ch.5] for a leisurely overview of Apéry's approach.

At the heart of Apéry's argument lie two magical sequences:

$$a_n: \ 1, 5, 73, 1445, \ldots, \qquad b_n: \ 0, 6, \frac{351}{4}, \frac{62531}{6}, \ldots,$$

both described by the recurrence relation

$$(n+1)^3 x_{n+1} = (34n^3 + 51n^2 + 27n + 5)x_n - n^3 x_{n-1}. \tag{12}$$

It appears that the sequence a_n, nowadays called the Apéry sequence, is integral–a fact which Apéry states without proof. This is a very unlikely assertion since the recursion (12) requires, at every step, division by $(n+1)^3$ in order to solve for the next term a_{n+1}. The second sequence b_n is no longer integral, yet Apéry claims that its denominators are arithmetically controlled in the sense that $d_n^3 b_n$ is integral, where $d_n = \mathrm{lcm}(1, 2, \ldots, n)$. The last piece of Apéry's puzzle is that

$$\frac{b_n}{a_n} \to \zeta(3).$$

The whole argument hinges on the fact that $(\sqrt{2}+1)^4 > e^3$. The greater number is the largest root of the quadratic equation $x^2 = 34x - 1$, the characteristic equation of (12), and it represents the exponential growth rate of the sequences a_n and b_n. The lesser number controls the exponential growth rate of the denominators in the sequence b_n.

In [6], Apéry also outlines a similar, but marginally simpler, sequential blueprint for proving the irrationality of $\zeta(2)$.

Abdelaziz [1] suggests that Apéry's insights come from extensive numerical experimentation. As such, Apéry's result really belongs to experimental mathematics; only later it was proved in a formal way.

6.2. **The man.** It seems fair to say that Apéry was, at the time of his discovery, somewhat of an outcast in French mathematics. Apéry, who had been educated in the classical Italianate style of algebraic geometry, was strongly against the formal French style of algebraic geometry that became mainstream in the 1960s under Grothendieck's influence. His mathematical interests soon drifted away from algebraic geometry to diophantine equations. In the period leading towards the late 1970s, French mathematics at large had become increasingly dominated by

Bourbaki–its books and its ideology, its former and current members and their students. Apéry's constructivist philosophy placed him on the fringes of the mathematical establishment at the time [3].

There are, however, interesting nuances to this somewhat simplistic picture. It is quite paradoxical that Apéry, a constructivist, was one of the first French academics to teach and promote category theory; see [2] for an interesting discussion. Also, despite his fervent anti-bourbakist views, Apéry maintained a lifelong and friendly relation with Jean Dieudonné, a Bourbaki founder and prophet. And in Caen, where he spent his entire academic career, Apéry enjoyed a great deal of local influence. It may then be a fitting description to call Apéry a 'mandarin de province'. Somewhere along stark differences in philosophy also runs a rivulet of mutual disdain between Paris and Province.

Apéry's complex and colorful personality is outlined by Mendès-France [15] with the following strong words:

> *C'est un personnage tonitruant, chaleureux, original, dictatorial, frisant parfois le ridicule, souvent attachant, pour tout dire ubuesque.*

According to Abdelaziz [1], Hendrik Lenstra had the following impression of Apéry:

> *... he was the local boss of mathematics [...] he behaved as if he was important, ordered secretaries around like a dictator. [...] He didn't create the impression he was serious.*

Upon hearing that Apéry had proved the irrationality of $\zeta(3)$, Lendstra's first reaction was "This clown?!".

An integral aspect of Apéry's personality was his political activism. In a moving biographical text [5] François Apéry, his son, summarizes it as follows:

> *With the same hearty resolution he had stood up for free thinking against clericalism, for radicalism against the Right in 1934, for Resistance against National Socialism in 1940, for the Republic against Gaullism in 1958, for constructivism against Bourbakism, for the university against leftism in 1968.*

In 1970 Apéry was decorated with the Légion d'Honneur for his involvement in the resistance against nazism under the Occupation, and then in the resistance against leftism. These two kinds of resistance are very different, and they happened nearly thirty years apart, yet they belong together in Apéry's political outlook.

[3] Much has been written about the Bourbaki school and its influence, in France, the United States, and elsewhere. Yet precious little, it seems, has been said about the anti-bourbakist stance–not now, when such a position is quite effortless, but then, at the time of the Bourbaki conformism. René Thom, in France, and John von Neumann, in the United States, were notable heretics. A forceful anti-bourbakist statement is that of Arnaud Denjoy in a 1954 letter to Henri Cartan, one of the Bourbaki founders:

> *Je redoute votre absolutisme, votre certitude de détenir la vraie foi en mathématiques, votre geste mécanique de tirer le glaive pour exterminer l'infidèle au Coran bourbakiste. [...] Nous sommes nombreux à vous juger despotique, capricieux, sectaire.*

Apéry's discovery of the irrationality of $\zeta(3)$ is his greatest mathematical legacy. He achieved it at the age of 61, just as he had been diagnosed with Parkinson's disease.

6.3. **The lecture.** Apéry presented his discovery in June 1978, at the Journées arithmétiques conference in Marseille. The lecture was extraordinary: the result and the approach were astounding, while the delivery was outlandish.

The bizarre lecture is vividly described by Mendès-France [15] who, it has to be said, was not actually present at the conference. The account is not only second-hand, but maybe also somewhat hyperbolical.

> *Lorsque, quelques mois avant les Journées arithmétiques de Marseille, le bruit courait que R. Apéry avait montré l'irrationalité de $\zeta(3)$, personne n'y croyait. Tout au plus un canular, même pas drôle. Aussi, pour la conférence de R. Apéry, la salle fut pleine et de curieux, et de "farceurs" venus faire du chahut. R. Apéry n'est pas un orateur et n'a pu ni calmer les esprits ni convaincre les auditeurs de sérieux de ses travaux. Comment un mathématicien non admiré de ses collègues pourrait-il avoir quelque chose de raisonnable à présenter sur un problème réputé insoluble? Son exposé, volontairement provocant, fut interrompu de bavardages, modulé par le brouhaha, ponctué de rires. Une vraie foire.*

Alfred van der Poorten was in the audience. Here is his understated report [17]:

> *Though there had been earlier rumours of his claiming a proof, scepticism was general. The lecture tended to strengthen this view to rank disbelief.*

Frits Beukers, another participant, describes Apéry's lecture in [9]:

> *The lecture did not look like a math lecture at all. If anything, it was a direct confrontation between Apéry and his French colleagues, with Apéry provoking his audience. The lecture was so messy and ununderstandable that it seemed nothing would come from it.*

Beukers also records the following exchange: when asked where the sequences a_n and b_n come from, Apéry replied that they grow in his back garden. The account of Beukers suggests that Apéry's lecture was intentionally provocative.

All in all, it is as if irrationality was not just the subject matter of the lecture, but also the chosen style. Yet one person in the audience, Henri Cohen, saw some truth in Apéry's implausible exposition. With help from Alfred van der Poorten and Don Zagier, Cohen managed to iron out the details of Apéry's approach–just in time for a well received lecture at the International Congress of Mathematicians, in Helsinki, in August 1978. Apéry's proof had finally been accepted. Cohen's account [10] appeared in print shortly after.

6.4. **Après Apéry.** André-Jeannin [4] has adapted Apéry's approach in order to show that certain series such as

$$\sum_{n=1}^{\infty} \frac{1}{F_n}$$

where F_n is the Fibonacci sequence, are irrational.

Concerning the irrationality of $\zeta(5), \zeta(7), \dots$, two results are worth mentioning. Rivoal [18] has shown the following:

Theorem 6.1 (Rivoal)**.** *Infinitely many ζ-values at odd integers are irrational.*

In fact, a quantitative statement holds [7, Thm.1]: if $k \geq 3$ is an odd integer, then the dimension of the $\mathbb{Q}$-vector space generated by $1, \zeta(3), \zeta(5), \dots, \zeta(k)$ is at least $(\ln k)/3$. It follows that there is an infinite subset of $\{1, \zeta(3), \zeta(5), \zeta(7), \dots\}$ which is linearly independent over $\mathbb{Q}$; in particular, at most one element of the infinite subset is rational, whence the previous theorem. A reasonable conjecture is that $\{1, \zeta(3), \zeta(5), \zeta(7), \dots\}$ itself is linearly independent over $\mathbb{Q}$.

A tangible irrationality result is due to Zudilin [21]:

Theorem 6.2 (Zudilin)**.** *One of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational.*

The question of *why* Apéry's approach works has also received some learned attention, see Zagier [20, Sec.1].

And yet, nearly 50 years later, no result comes close to matching Apéry's discovery.

Afterword

This essay is partly based on an undergraduate project that I wrote, back in 2002, for a number theory course at McGill University. I have a certain fondness for that project, as it was my first mathematical text. The impetus for revisiting it came about as I needed a way to channel my irritation with Nahin's book [16]. Subsequently, I was enthralled by the vibrant account of Apéry fils [5] and the illuminating text of Pierre Ageron [2]. Neither one contains mathematical proofs, yet the stories they tell are deeply mathematical.

I am grateful to Pierre Ageron for thoughtful comments and corrections on some of the points discussed in Section 6, and for sending me a copy of [15]. This entertaining account by Michel Mendès-France is quite hard to find. Pierre Ageron has also pointed out the unpublished thesis [1]. Youssef Abdelaziz has kindly agreed to share it, and I thank him for that–it is a very informative read.

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Department of Mathematical Sciences
Indiana University Indianapolis
Email address: `bnica@iu.edu`