# Euler and Hamilton Paths

Katrina Hoetmer

IUPUI

The Mathematical Life · April 2023

The Kőingsberg Bridges

– In 1700's Prussia, couples liked to go on walks and see if they could solve the bridge problem.

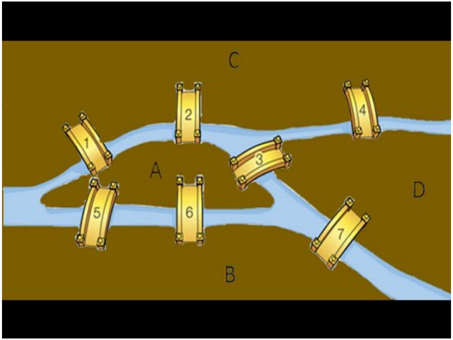


Figure: The bridges of Kőnigsberg

Now what is this bridge problem you may ask.

- There were 7 bridges in the city of Kőnigsberg.
- Couples would try to pass over every bridge exactly once on their strolls through town
- No one had ever been able to complete it and many deemed it was impossible
- This attracted the attention of Euler

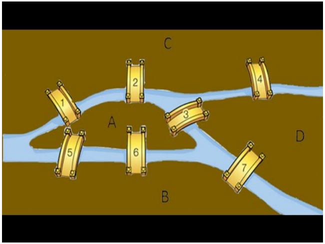


Figure: The bridges of Kőnigsberg

Euler generalized the problem and came up with the graph

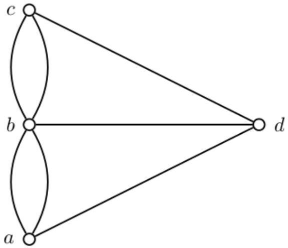


Figure: graph of the Konigsberg bridges

Where each edge represents a bridge and each vertex a land mass

Euler paths are defined as paths that use every edge in the graph exactly once. Today, we know several things about this type of problem. But, first we need some background.

- A graph is connected if you can trace a path between any 2 verices.
- A path is closed if it starts and ends at the same point
- How many edges are attached to a vertex is its degree
- A vertex is even if its degree is even, similarly with odd

What we know:

- Euler paths only exist for connected graphs
- A graph has a closed Euler path iff every vertex is even
- A graph has a open Euler path iff there are exactly 2 odd vertices

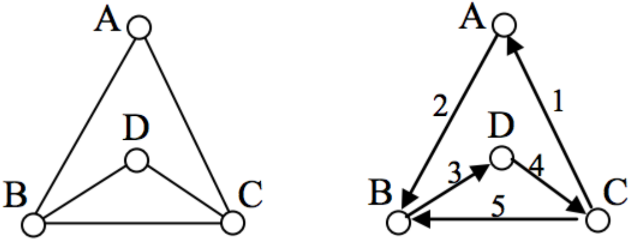


Figure: Example of a Euler path

In 1859, William Hamilton found himself short on drinking money, so he made a plan

- Invent a puzzle from a graph to draw people in
  - He called it "Voyage Around the World" and labeled each vertex with a major cities name
- Challenge people to find a way to visit each city in the ’world’ exactly once
- Gather drinking money from the people that tried


Figure: Hamilton’s Voyage Around the World

What is a Hamilton Path?

- A Hamilton path is a path that passes through each vertex in a graph exactly once
  - The first and last vertex can be the same

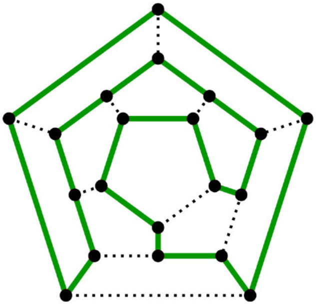

Figure: Example of Hamilton Path for Voyage Around the World

Not a lot is known about Hamilton paths. What we do know is this:

- A graph has an open Hamilton path if the sum of degrees of each pair of verticies of the graph is at least v-1
- A graph has a closed Hamilton path if the sum of degrees of each pair of verticies of the graph is at least v

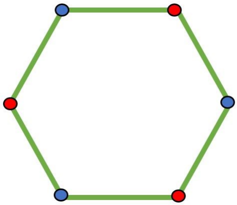

Figure: Example

What we know is not very helpful. Here is why:

- They are not necessary conditions for a Hamilton path to exist
  - As we saw in the example, $C_6$ doesn't meet the criteria for a Hamilton path, but it has both closed and open Hamilton paths
  - This is true for many graphs with Hamilton paths
- They are obvious
  - What we know really boils down to if the graph has a great number of evenly distributed edges then it has a Hamilton path
- Its hard to use
  - If your graph has even as little as 20 vertices, 190 pairs need to be computed
  - As the number of vertices grow the time to compute whether it has a Hamilton path grows exponentially

Hamilton paths are used in many real world situations. One that is studied extensively in Computer Science is the traveling salesman problem.

> The traveling salesman problem: A salesman wants to visit each town exactly once then return home. He wants to find the shortest route to accomplish this.

- This problem can also be done with weighted paths
  - For example they can be weighted by traffic data or speed limit

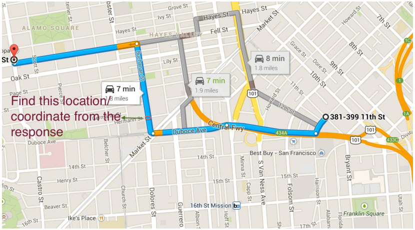


Figure: Google maps uses a weighted version of the traveling salesman problem

- Wikipedia: https://en.wikipedia.org/wiki/Travelling_salesman_problem
- Richard J Trudeau: Euler Walks and Hamilton Walks, Introduction to Graph Theory: 172–188, 1993