Peter Hilton

The Mathematical Component of a Good Education

Dedicated to Dr. Heinz Götze,
a truly educated man,
in respect and affection

1. Introduction

The main thesis of this article is that mathematics is, like music, worth doing for its own sake. This is, of course, not to deny the great usefulness of mathematics in pure and applied science; indeed, it was never more useful than it is today, when so many areas of human enquiry hitherto immune to mathematical contamination now find mathematics an essential tool in achieving progress and an essential language for the expression of their pertinent concepts and results.

However, the usefulness of mathematics, which is a theme receiving perfectly adequate attention in influential circles today, is not the theme of this article, except insofar as it tends to conceal and disguise the cultural aspect of mathematics. The role of music suffers no such distortion, for it is clearly an art whose exercise enriches composer, performer and audience; and music does not need to be justified by its contribution to some other aspect of human existence. Nobody asks, after listening to a Beethoven symphony, 'What is the use of that?'. The usefulness of mathematics, while perhaps contributing to the relative affluence of its skilled practitioners, does have very deleterious effects on the nature of mathematical education, which it would be well to describe quite explicitly. However, before detailing these distortions, it must be stated quite unequivocally that there is no gain for the utility of mathematics in committing them – on the contrary, for an appreciation of mathematics and an understanding of its inherent dynamic are necessary in order to be able to apply it effectively.

The first serious error is the confusion of education with training. This error, of course, goes far beyond mathematics education[1] –

[1] It is a wry commentary on the value-system in the United States that one speaks there of 'teacher training' and 'driver education'!

M. Atiyah et al., *Miscellanea Mathematica*

our bureaucrats and politicians now use the two terms quite synonymously – but it is particularly meretricious when applied to mathematics. For students, and their parents, believe that mathematics education should consist exclusively of the acquisition of a set of skills which prove useful in their later careers; so the skills must be learnt, that is, committed to memory, and no real understanding need occur. Of course, we cannot, in fact, predict what skills the student will need. What we can predict is that those skills will change and that the student will need to understand and not merely to remember. Adaptability to change is itself a hallmark of successful education, and it is change, not technology, which most aptly characterizes life today and in the foreseeable future. A genuine education enables one to acquire, for oneself, the skills one happens, at a given stage of one's life, to need. A training, on its own, contributes almost nothing to education and produces distressingly ephemeral advantages. Unfortunately some of the most influential formers of opinion in the English-speaking world – notably, Ronald Reagan and Margaret Thatcher – are vociferous advocates of the view that it is the function of our educational institutions to train, and that the success of their mission can therefore be measured by instruments appropriate to the market-place, employing the criteria of cost-benefit analysis.

The usefulness of mathematics leads to other, related abuses. Since mathematics is useful its acquisition must be tested. Since, in the perverse view we are deprecating, it is a skill, it is tested as a skill. Since it is useful, it must be taught to all. Thus the testing problem becomes enormous, and grading by machine becomes commonplace. The result is that the standard tests have almost nothing to do with the acquisition of mathematical understanding and put a premium on brute knowledge and memory, speed and slickness. They provide no opportunity for the student to explain his or her answer and treat all 'wrong' answers as equally wrong. Thus their effect is to distort the teaching and learning processes and the curriculum, in the direction of unmitigated skill-acquisition. They are, in short, inimical to mathematics itself. Let me be quite explicit and unequivocal – a testing procedure which gives the student no opportunity to explain his, or her, answer should have no place in an enlightened curriculum.

Further, the study of mathematics starts with the teaching of arithmetic, a horrible, wretched subject, far removed from real mathematics, but perceived to be useful. So vast numbers of intelligent people become 'mathematics avoiders' although they have never met mathematics. Their desire to avoid the tedium of elementary arithmetic, with its boring, unappetising algorithms and pointless drill-calculations, is perfectly natural and healthy (despite the practice of referring to mathematics avoidance in the language of pathology – 'mathophobia', 'math clinics', 'math anxiety'). Arithmetic is the cholesterol of elementary education, clogging the arteries of learning; if there ever was any justification for inflicting this unpalatable diet on our innocent children, there is none today, with the increasingly ubiquitous availability of low-priced hand-calculators. Yet, as we say, the practice continues, so that very few 'educated' people even understand what mathematics is, let alone have an appreciation of its potential role in enriching their lives and their culture[2].

Thus, to some, it must seem absurd to liken mathematics to music as an art to be savoured and enjoyed even in one's leisure time. Yet that is how it should appear and could appear if it were playing its proper role in our (otherwise) civilized society. Just as an appreciation of music is a hallmark of the educated person, so should be an appreciation of mathematics.

2. An Educated Person

There is, we claim, a valid and valuable concept of an educated person. The ancient Greeks had this concept, and it included for them an appreciation of mathematics, especially geometry; on the other hand, the Romans, conspicuously, did not. As Philip Howard writes, reporting on the 1989 meeting of the British Classical Association: 'The Romans were bad at science. They were practical men who followed intellectual pursuits only if they were useful and profitable, or, in the uncharming vogue phrase, "bankable skills". It is an attitude that is still with us'. Indeed it is! How comfortable

[2] 'And what will you make, when you are queen?'
It was silly, this, really – I mean, if any of one's friends could hear ...
'No more maths'
'Ah. That is difficult for the banks and the shops and the men of business. Never mind, we arrange'.

Penelope Lively, The French Exchange.

Mrs. Thatcher would have been in Roman society – if only it had accorded to women access to political dominance.

The broader concept of education was certainly again current in the 17th and 18th centuries in Britain and animated those who founded the Royal Society of London; other nations, too, in Europe and elsewhere in the world, have had their Enlightenments, their Renaissances. However, the concept began to undergo a curious transition in Victorian England. Certainly, it continued to connote the desire and the ability to go on learning, by reading and other forms of study; and it implied a familiarity with, and appreciation for, poetry, literature, music, the arts and architecture. It suggested a philosophical, reflective turn of mind. However, when the transition was complete, it carried two rather unfortunate connotations as well. The term tended to be applied to members of the leisured class (and, naturally but sadly, predominantly to the masculine sex); and there was no implication of a knowledge or appreciation of science.

This last feature largely persists to this day in Britain. Exasperation with its manifestations led C.P. Snow to deliver his celebrated Rede Lecture, 'The Two Cultures', in which he deplored the prevalence, in positions of prominence and influence, of people having no knowledge or understanding of the Second Law of Thermodynamics. Of course, Snow was not the first to remark on this phenomenon, but his own popularity as a writer and reputation as a thinker and man of affairs undoubtedly broadened the discussion, if it did not always succeed in deepening it. It is important to recall that Snow's viewpoint that a person was only to be considered educated if he or she was versed in the arts *and* the sciences was by no means universally accepted at the time his lecture was delivered and published. However, we believe that today the problem of acceptance is very different in certain contemporary societies where cultural philistinism is rampant. Now it is often necessary to argue that a technologically advanced society needs people with an understanding of history, an appreciation of language (theirs and other people's) and an awareness of the 'higher purposes' to which their increased affluence and computerized efficiency give them access. The proneness, to which we have already drawn attention, to confuse education with training has led, at least in the English-speaking world, to a marked down-grading of the study of the arts and the humanities, and to the emergence of the dangerous illusion that a modern industrial society should encourage applied science at the expense of pure science. Such an attitude, had it been widespread 20 years ago, would, for example, have seriously impeded the development of 'medium temperature' superconductors. It is surely clear, moreover, that an educated person should have some understanding of both pure and applied aspects of science. If, for example, he or she is is to appreciate the actual and potential roles of the computer in our and future societies, then the educated person must appreciate and comprehend significant parts of science, technology, logic and mathematics.

3. What is Mathematics?

Let me at this stage abandon, at least for the time being, my role of Cassandra and discuss briefly the essential qualifications of the educated person. Such a person should, of course, have all the traditional qualifications. In addition such a person must bestride 'The Two Cultures' understanding that both are vital to the individual and to society. With a better and deeper understanding of other civilizations, separated by time or space, or both, from our own, we are the more likely to take a more long-term view of our fundamental purposes on this earth, thus avoiding the glaring errors that arise today from short-term greed allied to technological skill and ignorant pride. However, it is my special case that mathematics is common to the 'two cultures', and the educated person should appreciate it.

It is not reasonable to expect lay persons to understand the details of sophisticated mathematical reasoning. Nevertheless, enough has surely been said to imply that our educated person must appreciate the role of mathematics in science and technology. Richard Feynman, echoing the thought of Galileo, has said: 'Nature talks to us in the language of mathematics', and it behoves educated people to understand just what this profound aphorism implies. Certainly such an understanding cannot be achieved without a far better insight into what mathematics actually is than is commonly found even among university-trained people today. Yet even such an insight, however essential, would not, in my opinion, be adequate; for mathematics grows and develops in many ways unrelated to science, and thus plays a crucial role in the history of human thought. So I argue that the educated person must understand what mathematics is – but not in the sense of a dictionary definition. Such a person must have an appreciation of mathematical reasoning and of the role of mathematics in the evolution and development of human society. Such an appreciation requires one to understand something of what mathematicians *do* – this would provide a much better working description of what mathematics is, in practice, than any dictionary could be expected to provide[3].

Unfortunately, as we will argue below, very few people have this kind of appreciation of the true nature of mathematics. The most common fallacy, even among otherwise well-informed people, is, as we have said, to confuse mathematics with elementary arithmetic, and to suppose that progress in mathematics consists of performing ever more complicated calculations with speed, dexterity and accuracy. Thus, for example, Dustin Hoffman received an Oscar in 1989 for his portrayal of the autistic brother in the film 'The Rain Man'. This person is an 'idiot savant', capable of performing rapid, totally unmotivated mental calculations such as 341×127, $\sqrt{19}$ to 10 decimal places. However, he is described by various critics, in their reviews of the film, as a genius. It is surely unnecessary for me to belabour the point further that such an extraordinary ability, far from being evidence of genius, is usually an indication of stupidity – as in this case. There have been rare exceptions, such as Gauss, the British civil engineer George Parker Bidder, and the Scottish statistician A.C. Aitken. However, it is interesting and significant to note that Gauss' powers of mental calculation declined as his genius grew, thus testifying to the antithesis between calculation and mathematical insight which we are claiming to exist.

In real life a characteristic example of the idiot savant was the Derbyshire agricultural labourer Jedediah Buxton, who was able to demonstrate that the Fermat number $2^{2^5}+1$ is not prime by actually factorizing it when it was given to him in decimal notation. He performed this feat in his head while carrying out his everyday duties. Buxton was brought to London to be examined by a group of Fellows of the Royal Society. He was taken to the theatre to see Garrick perform, to see how he would react to the experience.

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[3] Bertrand Russell's famous dictum that 'mathematics is the subject in which you don't know what you're talking about, and don't care whether what you say is true' is merely a philosophical joke, though a good one!

He reacted by compulsively counting the number of steps Garrick took during the performance! Thus indeed did Buxton symbolically demonstrate that arithmetical skill, of however high an order, is no part of our culture.

This justified conviction, on the part of many sensitive and 'educated' people, that arithmetic cannot be regarded as a part of the individual's cultural equipment, together with the erroneous belief that arithmetic is the essence of mathematics, has led to the widely-held view that mathematics itself is not to be regarded as a component of a liberal education. Thus many aesthetes are to be found positively glorying in their ignorance of, and ineptitude in, mathematics. Such people may proudly announce that they do not understand railway timetables, and are merely vexed by their difficulty in computing the tip in a restaurant. There are not to be found educated people who glory in their inability to use their language[4] or to read properly; anybody with such a difficulty would doubtless seek to conceal it.

Genuine mathematics, then, its methods and its concepts, by contrast with soulless calculation, constitute one of the finest expressions of the human spirit. The great areas of mathematics – algebra, real analysis, complex analysis, number theory, combinatorics, probability theory, statistics, topology, geometry, and so on – have undoubtedly arisen from our experience of the world around us, in order to systematize that experience, to give it order and coherence, and thereby to enable us to predict and perhaps control future events. However, within each of these areas, and between these areas, progress is very often made with no reference to the real world, but in response to what might be called the mathematician's apprehension of the natural dynamic of mathematics itself. Let us develop this theme further.

Mathematics, while essential to science, as Feynman has so vividly testified, has its own internal dynamic, powerful and subtle. Often, and today most especially, mathematics moves forward not under the stimulus of science but under the stimulus of its own recent advances. Applied mathematicians will often find a piece

[4] Regrettably, statistical evidence is accumulating to indicate that students in the United States, offered training rather than education, and fascinated by the potential of modern technology, are increasingly unable to use the English language properly to convey their ideas.

of mathematics, *developed for its own sake*, the precise tool they need for the expression and elucidation of their scientific problem.

All this is commonplace to the mathematician – but it is not for the research mathematicians that I write. Thus I will allow myself to offer a schema for the expert approach to a scientific problem. We may represent it diagrammatically as follows:

$$\begin{tikzcd} \boxed{\text{Real-world problem}} \arrow[r] & \boxed{\text{Scientific model}} \arrow[r] & \boxed{\text{Mathematical model}} \arrow[d, leftrightarrow] \\ & & \boxed{\text{Computer}} \end{tikzcd}$$

Modeling in Applied Mathematics

Thus, one first constructs a scientific model of the real-world situation one is studying. If the problem requires a physical model, then, typically, the scientific model will be concerned with physical entities (not necessarily observed or observable, perhaps merely postulated) and physical laws (for example, laws of conservation of energy and momentum). One then constructs a mathematical model to reason about the scientific model (for example, a differential equation such as the Navier-Stokes equation or the KdV equation, or an infinite-dimensional Lie algebra in superstring theory). One does calculations, perhaps based on experiment, in special cases and feeds the information back into the mathematical model to assist one in formulating plausible hypotheses and conjecturing general solutions.

Of course, this is a gross oversimplification of the process, which is nothing like as linear as our diagram above suggests. One may well refer back to the real world at some stage and decide that it is necessary to modify the scientific model or the mathematical model. Also one may make a mathematical model of the mathematical model, that is, one may embed the mathematical model in a broader class of mathematical problems for which there already exists a rather substantial theory; for example, the qualitative theory of differential equations of certain types may lead one into the generalization which consists of the study of vector fields on manifolds.

One feature of our modeling schema to which I wish to draw attention is that, with fairly minor modifications, it may be applied to 'pure' mathematics also. In that case the problem comes, of course, from within mathematics. To solve it, however, one may well need, or wish, to generalize the problem in order to be able to apply an existing theory or in order to have a theoretical framework within which to develop a solution. One may also conduct 'experiments', in the sense of considering special cases obtained by specifying certain variables or by simplifying the model (without distorting it); in the first case, it may well be necessary to do some well-motivated calculations.

Thus it emerges that there is no great difference between the procedures of pure and applied mathematics – there is really only one mathematics. Of course there is the difference that the source of the problem comes in one case from mathematics itself and in the other from the real world; but even here this difference is confined to the *original* source of the problem – the applied mathematician grappling with a differential equation is, at that point, behaving in a manner indistinguishable from that of a pure mathematician.

Indeed, to strike a controversial note, it could be argued that the pure mathematician has opportunities for application which transcend those of the applied mathematician. One can apply mathematics to solve problems in physics – but it is difficult (though not, perhaps, absolutely impossible) to conceive of applying physics to solve problems in mathematics. However, within mathematics, it is perfectly clear, indeed commonplace, that one may, for example, apply algebra to solve a problem in geometry, or apply geometry to solve a problem in algebra.

The foregoing discussion is designed to show, in outline, what mathematics is. My own position, as a mathematician, is to be suitably humble about my own contributions to mathematics, but not modest at all about my claims for mathematics itself. This was the position adopted by my teacher and friend, the great British topologist Henry Whitehead – though he had far less justification for his humility! Whitehead argued that there are relatively few pursuits in life which are inherently worth while – he instanced the making of music and the design of elegant and useful furniture – and that doing, or at least appreciating, mathematics is one of them. It is surely reasonable to equate Whitehead's concept of intrinsically valuable pursuits with our own concept of the desiderata of the educated person. There is, in fact, no doubt in my mind that mathematical appreciation is not only a component part of the education of civilized people, but a pillar of that education. I long for the day when, indeed, mathematics will be appreciated and enjoyed by educated laymen as an art and also respected as the mainstay of science. It has been so in the past, but it is not so now. Is it too optimistic to hope it might be so again?