ON THE PEKERIS MODES OF THE ORR-SOMMERFELD PROBLEM FOR PLANE POISEUILLE FLOW

by

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SUMMARY

The Pekeris modes are damped modes of the 'centre' or 'fast' type for which $c_r \uparrow 1$ and $c_i \uparrow 0$ as $\alpha R \to \infty$. The numerical results obtained by Orszag (1) for $\alpha = 1$ and $R = 10\,000$ showed that the eigenvalues for the even and odd modes of this type are very nearly equal, and one of the goals of this paper is to provide an analytical explanation for this rather striking result. Under certain simplifying assumptions we are led to a fourth-order equation which can be viewed as a generalization of Weber's equation for the parabolic cylinder functions, and the eigenvalue problem is then posed on an infinite interval. An asymptotic analysis of this problem shows that the eigenvalue relations for the even and odd modes are indeed the same even though the underlying analysis is significantly different in the two cases. Explicit results are also given for both the even and odd eigenfunctions. The even eigenfunctions are similar to the Whittaker functions $D_n(x)$ but the odd eigenfunctions involve Dawson's integral and certain polynomials.
1. Introduction

The stability of plane Poiseuille flow is governed by the equation

\[(i\alpha R)^{-1}(D^2 - \alpha^2)^2 \phi - (1 - z^2 - c)(D^2 - \alpha^2)\phi - 2\phi = 0,\]  

(1.1)

where \(\phi(z)e^{i\alpha(z-ct)}\) is the stream function in the usual normal mode analysis, \(R\) is the Reynolds number, and \(D = d/dz\). We also have the boundary conditions

\[\phi = D\phi = 0 \quad \text{at} \quad z = \pm 1.\]  

(1.2)

In the usual temporal stability problem, in which the growth or decay of a disturbance in time is considered, we take \(\alpha\) and \(R\) to be real and then treat \(c\) as the eigenvalue parameter.

Historically the main interest in this eigenvalue problem has focused on the one (even) mode which exhibits a weak viscous instability for \(R \gtrsim 5772\). But there is another aspect of the problem which is also of considerable interest, namely the distribution of the eigenvalues \(c\) in the \((c_r, c_i)\)-plane either for fixed values of \(\alpha\) and \(R\) or for bounded values of \(\alpha\) as \(\alpha R \to \infty\). The numerical results obtained by Orszag (1) and Mack (2) for \(\alpha = 1\) and \(R = 10000\) show that there are three distinct families of eigenvalues which, in Mack’s classification, are designated by the letters A, P, and S in honour of G. B. Airy, C. L. Pekeris, and I. V. Schensted.

In this paper we will consider only the Pekeris modes. These are stable modes of the ‘centre’ or ‘fast’ type for which \(c_r \uparrow 1\) and \(c_i \uparrow 0\) as \(\alpha R \to \infty\). For the even modes of this type Pekeris (3) found that

\[c \sim 1 - (4n + 5)e^{\frac{1}{2}n\pi i(\alpha R)^{-\frac{1}{2}}} \quad (n = 0, 1, 2, \ldots)\]  

(1.3)
and some details of his analysis will be given later in section 3. Orszag’s results (1) also show that the eigenvalues for the even and odd modes are very nearly equal and one of the goals of this paper is to provide an analytical explanation for this near equality.

2. The eigenvalue problem

We begin by noting that the turning points of equation (1.1) are located at $z = (1 - c)^{1/2}$ and that when $c$ has the behaviour (1.3) they coalesce as $\alpha R \to \infty$. This suggests that we make the change of variables

$$\phi(z) = \psi(x),$$

where

$$x = (2\lambda)^{1/2} e^{-\frac{1}{4} \pi i} z \quad \text{and} \quad \lambda = (\alpha R)^{1/2}. \quad (2.2)$$

Then, without approximation, equation (1.1) can be rewritten in the form

$$\psi'''' + (\mu - \frac{1}{4} x^2)\psi'' + \frac{1}{2} \psi = \delta \{ \psi'' + \frac{1}{2} (\mu - \frac{1}{4} x^2) \psi \} - \frac{1}{4} \delta^2 \psi, \quad (2.3)$$

where

$$\delta = \alpha^2 e^{\frac{1}{4} \pi i} \lambda^{-1} \quad \text{and} \quad \mu = \frac{1}{2} e^{-\frac{1}{4} \pi i} (1 - c) \lambda, \quad (2.4)$$

and the boundary conditions (1.2) become

$$\psi = \psi' = 0 \quad \text{at} \quad x = \pm (2\lambda)^{1/2} e^{-\frac{1}{4} \pi i}. \quad (2.5)$$

To simplify this eigenvalue problem we next make two important assumptions. First, following Pekeris, we assume that $\alpha^2 \ll 1$ so that the terms on the right-hand
side of (2.3) can be neglected. The governing equation thus becomes simply

\[ \psi^{iv} + (\mu - \frac{1}{4} x^2)\psi'' + \frac{1}{2} \psi = 0. \quad (2.6) \]

Second, we assume that \( \lambda \gg 1 \) so that the boundary conditions (2.5) can be replaced by the simpler condition

\[ \psi(x) \text{ bounded as } x \to \pm \infty e^{-\frac{1}{2}x^i}. \quad (2.7) \]

The problem is thus posed on an infinite interval along the ray \( \text{ph} \ x = -\frac{1}{8} \pi \). To simplify the analysis, however, we will formally treat \( x \) as a real variable except where noted otherwise. Equation (2.6) bears a certain similarity to Weber’s equation for the parabolic cylinder functions and some parts of the present discussion were motivated by known results for that equation (4).

For some purposes it is convenient to let \( \mu = 2\nu + \frac{5}{2} \) so that equation (2.6) becomes

\[ \psi^{iv} + (2\nu + \frac{5}{2} - \frac{1}{4} x^2)\psi'' + \frac{1}{2} \psi = 0. \quad (2.8) \]

For fixed \( \nu \) and large \( x \) we expect that this equation will have two solutions of dominant-recessive type, one solution of balanced type, and one solution of well-balanced type. First approximations to the solutions of dominant-recessive type can be obtained by neglecting the term \( \frac{1}{2} \psi \) in (2.8) and then using Olver’s results for Weber’s equation (5, pp.206-7). In this way we find that the asymptotic forms of the solutions of dominant-recessive type are constant multiples of

\[ x^{-2\nu - 5} e^{\frac{1}{4}x^2} \quad \text{and} \quad x^{2\nu} e^{-\frac{1}{4}x^2}. \quad (2.9) \]

First approximations to the solutions of balanced and well-balanced type can then be obtained by retaining only the terms \( -\frac{1}{4} x^2 \psi'' \) and \( \frac{1}{2} \psi \) in (2.8) and this immediately
gives the asymptotic forms
\[ x^2 \quad \text{and} \quad x^{-1}. \]  
(2.10)

We also note that equation (2.8) has one exact solution of well-balanced type, namely \( x^2 - 4(2\nu + \frac{5}{2}) \) for any value of \( \nu \).

Pekeris (3) noticed that equation (2.8) can be integrated once to yield
\[ L_\nu \psi \equiv \psi'''' + (2\nu + \frac{5}{2} - \frac{1}{4}x^2)\psi' + \frac{1}{2}x\psi = c_\nu \]  
(2.11)

and this equation provides the natural starting point for the present analysis. Pekeris considered only even solutions and so set \( c_\nu = 0 \). More generally, however, when \( c_\nu = 0 \) we have two solutions of dominant-recessive type which behave like (2.9) and one exact solution of well-balanced type. When \( c_\nu \neq 0 \), the particular integral of (2.11) is of balanced type and is asymptotic to \( \frac{4}{3}c_\nu x^{-1} \) for fixed \( \nu \) and large \( x \).

These remarks then suggest that we define four solutions of (2.11) according to the following scheme:

\[ \psi_\nu^{(0)}(x) \sim x^{2\nu} e^{-\frac{1}{4}x^2} \quad (c_\nu = 0; \text{recessive}), \]  
(2.12 a)

\[ \psi_\nu^{(1)}(x) \sim \frac{4}{3}c_\nu x^{-1} \quad (c_\nu \neq 0; \text{balanced}), \]  
(2.12 b)

\[ \psi_\nu^{(2)}(x) = x^2 - 4(2\nu + \frac{5}{2}) \quad (c_\nu = 0; \text{exact; well-balanced}), \]  
(2.12 c)

\[ \psi_\nu^{(3)}(x) \sim x^{-2\nu-5} e^{\frac{1}{4}x^2} \quad (c_\nu = 0; \text{dominant}). \]  
(2.12 d)
3. The eigenvalue relation for the even modes

Let us begin this discussion by giving a brief summary of Pekeris’s analysis for the even modes. He started from an equation which is equivalent to (2.11) with \( c_\nu = 0 \) and then let

\[
\psi(x) = \hat{\psi}(s), \quad \text{where} \quad s = \frac{1}{2} x^2.
\]  

(3.1)

Equation (2.11) then becomes

\[
s\hat{\psi}'''' + \frac{3}{2} \hat{\psi}'' + (\nu + \frac{5}{4} - \frac{1}{4} s)\hat{\psi}' + \frac{1}{4} \hat{\psi} = 0
\]

(3.2)

On differentiation of this equation we obtain a second-order equation for \( \hat{\psi}''(s) = y(s) \) (say) where

\[
sy'' + \frac{5}{2} y' + (\nu + \frac{5}{4} - \frac{1}{4} s)y = 0.
\]

(3.3)

One solution of this equation is

\[
y(s) = e^{-\frac{1}{2} s} {}_1 F_1 (-\nu; \frac{5}{2}; s);
\]

(3.4)

the other solution is singular at the origin and must therefore be rejected. For this solution to be bounded as \( s \to \infty \) we must have

\[
\nu = n \quad (n = 0, 1, 2, \ldots)
\]

(3.5)

and this is the required eigenvalue relation for the even modes. The corresponding eigenfunctions can then be found by integrating

\[
\hat{\psi}''(s) = e^{-\frac{1}{2} s} {}_1 F_1 (-n; \frac{5}{2}; s).
\]

(3.6)

Another approach to the eigenvalue problem for the even modes is to consider the asymptotic expansion of \( \psi_\nu^{(0)}(x) \) as \( x \to + \infty \). A direct calculation from (2.11) with
\(c_\nu = 0\) shows that

\[
\psi^{(0)}_\nu(x) \sim x^{2\nu} e^{-\frac{1}{4} x^2} \{ 1 - \nu(2\nu - 5)x^{-2} + \frac{1}{2}\nu(\nu - 1)(4\nu^2 - 24\nu + 51) x^{-4} - \ldots \}. \tag{3.7}
\]

This result suggests that when \(\nu = n = 0, 1, 2, \ldots\) the series in (3.7) terminates and we then have even eigenfunctions of the form

\[
\psi^{(0)}_n(x) = u_n(x) e^{-\frac{1}{4} x^2} \quad (n = 0, 1, 2, \ldots), \tag{3.8}
\]

where

\[
u_0(x) = 1, \quad \nu_1(x) = x^2 + 3, \quad \nu_2(x) = x^4 + 2x^2 + 19, \ldots \tag{3.9}
\]

More generally, as will be shown in section 5, it is possible to obtain an integral representation for the polynomials \(u_n(x)\) from which it is then found that they can be expressed as linear combinations of Hermite polynomials.

Neither of these approaches, however, is applicable to the odd modes. To deal with both the even and odd modes we now wish to describe an asymptotic approach based on the supposition that the parameter \(\mu\) is large. For the even modes we begin with the third-order homogeneous equation.

\[
\psi'''' + (\mu - \frac{1}{4} x^2) \psi' + \frac{1}{2} x \psi = 0. \tag{3.10}
\]

Next we make the change of variables

\[
\mu = (2\epsilon)^{-\frac{3}{2}} \quad \text{and} \quad x = (2/\epsilon^3)^{\frac{1}{2}} t \tag{3.11}
\]

so that equation (3.10) becomes

\[
e^3 \psi'''' - \frac{1}{2} (t^2 - 1) \psi' + t \psi = 0. \tag{3.12}
\]

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Now let $\Psi_1(t)$ denote the solution of (3.12) which is recessive as $t \to +\infty$. In the first instance we need to derive an approximation to $\Psi_1(t)$ which is uniformly valid when $t > -1$. Re-expansion of this approximation then provides the solution to the connection problem across the turning point at $t = 1$. The eigenvalue relation for the even modes is then obtained by imposing the symmetry conditions

$$\Psi'_1(0) = 0 \quad \text{and} \quad \Psi''_1(0) = 0. \quad (3.13)$$

From (3.12), however, we see that either of these conditions implies the other.

We begin by introducing the Langer variable $\eta(t)$ in the usual way by letting

$$\eta \eta' = \frac{1}{2}(t^2 - 1), \quad (3.14)$$

where the normalization has been fixed by requiring that $\eta'(1) = 1$. We then have

$$\frac{2}{3} \eta^3 = \int_1^t \left\{ \frac{1}{2}(v^2 - 1) \right\}^{1/2} dv \quad (t \geq 1)$$

$$= 2^{-3/2} \left\{ t(t^2 - 1)^{1/2} - \cosh^{-1} t \right\} \quad (3.15)$$

and

$$\frac{2}{3}(-\eta)^3 = \int_t^1 \left\{ \frac{1}{2}(1 - v^2) \right\}^{1/2} dv \quad (-1 < t \leq 1)$$

$$= 2^{-3/2} \left\{ \cosh^{-1} t - t(1 - t^2)^{1/2} \right\}. \quad (3.16)$$

For later use we note that

$$\frac{2}{3}(-\eta_0)^3 = \frac{1}{8} \sqrt{2\pi}, \quad \text{where} \quad \eta_0 = \eta(0). \quad (3.17)$$

The general asymptotic theory of the Orr-Sommerfeld equation has been discussed in great detail in (6) and that theory is directly applicable to equation (3.12). Some
simplification is possible, however, because we are treating \(x\) and \(t\) as real variables.

Thus, in terms of the generalized Airy functions defined in the Appendix, the ‘first approximation’ to \(\Psi_1(t)\) is given by

\[
\Psi_1(t) \sim A(t) \text{Ai}(\zeta, 2) + \epsilon^2 B(t) \text{Ai}(\zeta, 1) + \epsilon C(t) \text{Ai}(\zeta, 0),
\]  

(3.18)

where \(\zeta = \eta / \epsilon\) and where the slowly varying coefficients \(A(t), B(t), C(t)\) must all be analytic at \(t = 1\).

It is easy to show that \(A(t)\) satisfies the equation

\[
\eta' A' - 2 \eta'' A = 0
\]

(3.19)

and if we fix the normalization in the usual way by requiring that \(A(1) = 1\) then we have

\[
A(t) = \eta' = \frac{1}{2}(t^2 - 1)/\eta.
\]

(3.20)

Once \(A(t)\) is known, the easiest way of determining \(B(t)\) and \(C(t)\) is by matching the WKB approximation to \(\Psi_1(t)\) for \(t > 1\) to the outer expansion of (3.18) for \(\eta > 0\).

The required WKB approximation is easily found from (3.12) to be

\[
\Psi_1(t) = \frac{1}{2} \pi^{-\frac{1}{2}} \epsilon^{\frac{3}{4}} (\eta \epsilon')^{-\frac{3}{4}} \exp\left(-\frac{2}{3} \epsilon^{-\frac{3}{2}} \eta^{\frac{3}{2}}\right) \{1 - \epsilon^{\frac{3}{2}} H(t) + O(\epsilon^3)\},
\]

(3.21)

where

\[
H(t) = \sqrt{2}\left\{\frac{10}{24} t^3 (t^2 - 1)^{-\frac{3}{2}} - \frac{13}{4} t (t^2 - 1)^{-\frac{3}{2}}\right\}
\]

(3.22)

and the normalization has been fixed in a convenient way. We also note that

\[
H(t) \sim \frac{23}{24} \sqrt{2} \quad \text{as} \quad t \to +\infty
\]

(3.23)
and hence the approximation (3.21) remains uniformly valid as $t \to +\infty$. Thus, on matching (3.21) to the outer expansion of (3.18) we obtain (cf. 6, p. 284)

$$
\mathcal{A}(t) + \eta \mathcal{C}(t) = \eta^{\frac{5}{8}}
$$

(3.24)

and

$$
\frac{101}{48} \eta^{-\frac{3}{2}} \mathcal{A}(t) + \eta^{\frac{1}{2}} \mathcal{B}(t) + \frac{5}{48} \eta^{-\frac{1}{2}} \mathcal{C}(t) = \eta^{-\frac{5}{8}} H(t),
$$

(3.25)

and it is not difficult to verify that $\mathcal{B}(t)$ and $\mathcal{C}(t)$ are indeed analytic at $t = 1$.

Consider next the outer expansion of $\Psi_1(t)$ for $-1 < t < 1$. By using the asymptotic expansion of $\text{Ai}(-x, 2)$ which is given in the Appendix we find that

$$
\Psi_1(t) = \frac{1}{2} (1 - t^2) \varepsilon^{-1} + \pi^{-\frac{1}{2}} \varepsilon^{\frac{5}{8}} (-\eta \varepsilon^2)^{-\frac{5}{8}} \left\{ \sin \left[ \frac{2}{3} \left(-\zeta\frac{3}{2} - \frac{3}{4} \pi \right) \right]

- \varepsilon^{\frac{3}{2}} \hat{H}(t) \cos \left[ \frac{2}{3} \left(-\zeta\frac{3}{2} - \frac{3}{4} \pi \right) \right] + O(\varepsilon^3) \right\},
$$

(3.26)

where

$$
\hat{H}(t) = \sqrt{2} \left\{ \frac{101}{24} t^2 (1 - t^2)^{-\frac{3}{2}} + \frac{13}{4} t (1 - t^2)^{-\frac{1}{2}} \right\}.
$$

(3.27)

This result thus provides the solution to the connection problem for $\Psi_1(t)$ across the turning point at $t = 1$. There are two other aspects of this result which deserve special comment. First, the term $\frac{1}{2} (1 - t^2) \varepsilon^{-1}$ in (3.26) is a direct consequence of equation (3.12) being of third order. Second, the fact that $\hat{H}(t)$ is an odd function of $t$ will play a crucial role in our discussion of the eigenvalue relation for the even modes. From (3.26) we then have

$$
\Psi_1'(t) = - t \varepsilon^{-1} - \pi^{-\frac{1}{2}} \varepsilon^{-\frac{3}{8}} (-\eta \varepsilon^2)^{-\frac{3}{4}} \left\{ \cos \left[ \frac{2}{3} \left(-\zeta\frac{3}{2} - \frac{3}{4} \pi \right) \right]

+ \varepsilon^{\frac{3}{2}} \left[ \hat{H}(t) - \frac{5}{4} t (-\eta \varepsilon^2)^{-\frac{3}{2}} \right] \sin \left[ \frac{2}{3} \left(-\zeta\frac{3}{2} - \frac{3}{4} \pi \right) \right] + O(\varepsilon^3) \right\}
$$

(3.28)

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For an even mode we must require that $\Psi'_1(0) = 0$ and (3.28) then gives

$$\cos\left[\frac{2}{3}(-\zeta_0)\frac{3}{4} - \frac{3}{4}\pi\right] = 0$$

(3.29)

or

$$\frac{2}{3}(-\zeta_0)\frac{3}{4} - \frac{3}{4}\pi = (n + \frac{1}{2})\pi,$$

(3.30)

where $\zeta_0 = \zeta(0)$. Thus, on using (3.17), we obtain finally

$$\mu = 2n + \frac{5}{2} \quad (n = 0, 1, 2, \ldots),$$

(3.31)

where the enumeration of the eigenvalues has been fixed by an appeal to Pekeris’s result (3.5).

4. The eigenvalue relation for the odd modes

Let us return now to equation (2.11) with $c_r \neq 0$. If we make the change of variables (3.11) and set $c_r = (2e)^{-\frac{4}{3}}$ then we obtain

$$e^3\psi'' - \frac{\alpha}{2}(t^2 - 1)\psi' + t\psi = 1.$$  \hfill (4.1)

Now let $\Psi_2(t)$ denote the solution of this equation which behaves like $\frac{2}{3}t^{-1}$ as $t \to +\infty$. Note that this asymptotic behaviour is consistent with (2.12 b). As in the previous section, we must first derive an approximation to $\Psi_2(t)$ which is uniformly valid when $t > -1$. This can be done either by using the generalized Airy functions of $B$-type (6)
or more simply, when the variables are real, by using the generalized Scorer functions which are defined in the Appendix. Thus, in a ‘first approximation’ to \( \Psi_2(t) \), we have

\[
\Psi_2(t) \sim \epsilon^{-1} G(t) + A(t) \text{Gi}(\zeta, 2) + \epsilon^2 B(t) \text{Gi}(\zeta, 1) + \epsilon C(t) \text{Gi}(\zeta, 0),
\]

(4.2)

where \( A(t), B(t), C(t) \) are the same as in (3.18). Thus, we need only determine \( G(t) \) and that can also be done by a simple matching procedure. For that purpose consider first the so-called ‘singular inviscid solution’ \( \Psi_2^{(0)}(t) \), i.e. the solution of (4.1) when \( \epsilon = 0 \). This gives

\[
\Psi_2^{(0)}(t) = t - \frac{1}{2} (t^2 - 1) \ln\left\{ (t + 1)/(t - 1) \right\}
\]

(4.3)

which is automatically asymptotic to \( \frac{2}{3} t^{-1} \) as \( t \to +\infty \). We also need the outer expansion of \( \Psi_2(t) \) for \( t > 1 \). By using equation (A.9) and its derivatives we obtain

\[
\text{outer}\{\Psi_2(t)\} = \epsilon^{-1} G(t) + A(t) \left\{ \zeta \ln \zeta + \left[ \frac{1}{3} (\ln 3 + 2\gamma) - 1 \right] \zeta \right. \\
+ \frac{1}{2} 3^{-\frac{1}{3}} \Gamma\left( \frac{2}{3} \right) \right\} + O(\epsilon^2 \ln \epsilon).
\]

(4.4)

The required matching can then be posed in the form

\[
\text{outer}\{\Psi_2(t)\} \equiv \Psi_2^{(0)}(t) \quad \text{for} \quad t > 1
\]

(4.5)

from which we obtain

\[
G(t) = t - \frac{1}{2} (t^2 - 1) \left\{ \ln(t + 1) + \ln(\eta/(t - 1)) \right\} + \frac{1}{2} (t^2 - 1) \left\{ \ln \epsilon - \frac{1}{3} (\ln 3 + 2\gamma) - 1 \right\} - \epsilon \frac{1}{2} 3^{-\frac{1}{3}} \Gamma\left( \frac{2}{3} \right) A(t),
\]

(4.6)

and it is clearly evident that \( G(t) \) is indeed analytic at \( t = 1 \).
Consider next the outer expansion of $\Psi_2(t)$ for $-1 < t < 1$. For that purpose we first use the connection formula (A.11) to rewrite equation (4.2) in the form

$$
\Psi_2(t) \sim \epsilon^{-1} G(t) - \{ A(t) \text{Hi}(\zeta, 2) + \epsilon^2 B(t) \text{Hi}(\zeta, 1) + \epsilon C(t) \text{Hi}(\zeta, 0) \} 
\quad + \pi \{ A(t) \text{Bi}(\zeta, 2) + \epsilon^2 B(t) \text{Bi}(\zeta, 1) + \epsilon C(t) \text{Bi}(\zeta, 0) \}. \quad (4.7)
$$

The advantage of writing $\Psi_2(t)$ in this form is that when we take its outer expansion for $-1 < t < 1$ there is then a clear distinction between the oscillatory and non-oscillatory terms in the expansion. Thus, on using the asymptotic expansions of $\text{Bi}(-x, 2)$ and $\text{Hi}(-x, 2)$ which are given in the Appendix and after some remarkable cancellation, we find that the outer expansion of $\Psi_2(t)$ for $-1 < t < 1$ is given by

$$
\text{outer}\{\Psi_2(t)\} = \epsilon^{-1} \{ t + \frac{1}{2} (1 - t^2) \ln[(1 + t)/(1 - t)] \}
\quad + \pi \frac{1}{2} \epsilon \frac{\alpha}{2 \eta} \left( \frac{\beta}{\alpha^2} \right)^{-\frac{1}{2}} \left\{ \cos\left[ \frac{2}{3} (-\zeta)^{\frac{3}{2}} - \frac{3}{4} \pi \right] 
\quad + \epsilon^{\frac{3}{2}} \tilde{H}(t) \sin\left[ \frac{2}{3} (-\zeta)^{\frac{3}{2}} - \frac{3}{4} \pi \right] + O(\epsilon^3) \right\}. \quad (4.8)
$$

This result thus provides the solution to the connection problem for $\Psi_2(t)$ across the turning point at $t = 1$, i.e. a solution $\Psi_2(t)$ which behaves like (4.3) for $t > 1$ has the continuation (4.8) for $-1 < t < 1$.

For an odd mode we must require that $\Psi_2(0) = 0$ and (4.8) then shows that the eigenvalue relation for the odd modes is the same as for the even modes, namely (3.29).

Note that the same result is obtained if we had instead required that $\Psi_2''(0) = 0$. In this case, however, the enumeration of the eigenvalues cannot be fixed analytically and instead we must appeal to the numerical results given in Table 1.

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Thus, within the framework of the assumptions made in section 2, we conclude that the eigenvalues for the even and odd modes are indeed the same even though the underlying analysis in the two cases is significantly different.

5. The even eigenfunctions $\psi_n^{(0)}(x)$

In section 3 it was suggested that the even eigenfunctions are of the form

$$\psi_n^{(0)}(x) = u_n(x)e^{-\frac{1}{4}x^2} \quad (n = 0, 1, 2, \ldots), \quad (5.1)$$

where the $u_n(x)$ are polynomials of degree $2n$, and our goal in this section is to obtain a more systematic way of deriving these polynomials. For that purpose we begin with equation (3.10) in the form

$$\psi''' + (2n + \frac{5}{2} - \frac{1}{4}x^2)\psi' + \frac{1}{2}x\psi = 0, \quad (5.2)$$

where we have now set $\mu = 2n + \frac{5}{2}$. We remark in passing that this equation also has unbounded odd solutions and they will be discussed later in section 7.

For the bounded even solutions we now let

$$\psi(x) = u(x)e^{-\frac{1}{4}x^2}. \quad (5.3)$$

Equation (5.2) then shows that $u(x)$ must satisfy the equation

$$u'' - \frac{3}{2}xu'' + (2n + 1 + \frac{1}{2}x^2)u' - nuu = 0. \quad (5.4)$$
Note that this equation admits solutions which behave like $x^{2n}$ as $x \to \pm \infty$. Next we look for solutions of (5.4) in the form of a Laplace contour integral

$$u(x) = \int_C f(t) e^{xt} \, dt.$$  \hfill (5.5)

Equation (5.4) is then satisfied provided $f(t)$ is a solution of the equation

$$t f'' + \{2(n + 1) + 3t^2\} f' + \{4(n + 2) t + 2t^3\} f = 0$$  \hfill (5.6)

and $C$ is chosen so that

$$\left[ \{tf' + (2n + 1 - xt + 3t^2) f\} e^{xt}\right]_C = 0.$$  \hfill (5.7)

The solutions of (5.6) can be expressed in terms of confluent hypergeometric functions. One solution is simply the elementary function

$$f_1(t) = (2n + 3 - t^2) e^{-t^2}$$  \hfill (5.8)

and if $C$ is chosen to run from $-\infty$ to $+\infty$ then we recover the unbounded even solutions $\psi_n^{(2)}(x)$. The other solution of (5.6) can be expressed in the form

$$f_2(t) = t^{-2n-1} e^{-\frac{1}{2} t^2} \frac{1}{t^{2n+1}} \Gamma(2; \frac{1}{2} - n; -\frac{1}{2} t^2).$$  \hfill (5.9)

Thus we have

$$u_n(x) = \frac{(2n)!}{2\pi i} \int \frac{1}{t^{2n+1}} e^{xt-\frac{1}{2}t^2} \frac{1}{\Gamma(2; \frac{1}{2} - n; -\frac{1}{2} t^2)} \, dt,$$  \hfill (5.10)

where the normalization has been fixed so that the coefficient of $x^{2n}$ in $u_n(x)$ is one.

To evaluate the integral (5.10) recall that

$$e^{xt-\frac{1}{2}t^2} = \sum_{s=0}^{\infty} H_s(x) \frac{t^s}{s!},$$  \hfill (5.11)

where $H_s(x)$ are the Hermite polynomials in the notation of (7). The terms in the series (5.11) with $s = 1, 3, 5, \ldots$ make no contribution to $u_n(x)$ so we can write

$$u_n(x) = \frac{(2n)!}{2\pi i} \int \frac{1}{t^{2n+1}} \sum_{s=0}^{\infty} H_{2s}(x) \frac{t^{2s}}{(2s)!} \frac{1}{\Gamma(2; \frac{1}{2} - n; -\frac{1}{2} t^2)} \, dt.$$  \hfill (5.12)
Next we let
\[ \mathbf{1}_F(2; \frac{1}{2} - n; -\frac{1}{2} t^2) = \sum_{s=0}^{\infty} A_s(n) t^{2s}, \quad (5.13) \]
where
\[ A_s(n) = (-)^s \frac{1 + s}{2^s (\frac{1}{2} - n)_s}, \quad (5.14) \]
Thus on substituting (5.13) into (5.12) we obtain the desired result
\[ u_n(x) = (2n)! \sum_{s=0}^{n} \frac{1}{(2n - 2s)!} A_s(n) H e_{2n-2s} (x). \quad (5.15) \]
On setting \( n = 0, 1, 2 \) in this equation we immediately recover the results (3.9).

To illustrate the behaviour of these eigenfunctions consider \( \psi_0^{(0)}(x) = e^{-\frac{1}{2} x^2} \). In terms of the physical variables this becomes
\[ \phi_0^{(0)}(z) = \exp\{-\frac{1}{2}(\alpha R)^{\frac{1}{2}} e^{-\frac{1}{2} \pi i} z^2\}. \quad (5.16) \]
The real and imaginary parts of \( \phi_0^{(0)}(z) \) are shown in Fig. 1 for \( \alpha R = 10000 \). These analytical results are indistinguishable from the corresponding numerical solution simply because of their rapid exponential decay.

\[ \text{Fig.1 near here} \]
6. The odd eigenfunctions $\psi_{n}^{(1)}(x)$

One method of dealing with the solutions of this type is to derive an integral representation of Laplace type directly from equation (2.6). In this way it is found that the solutions can be expressed explicitly in terms of certain polynomials and Dawson’s integral

$$F(x) = e^{-x^2} \int_0^x e^{t^2} dt.$$  \hspace{1cm} (6.1)

For later reference we note that $F(x)$ satisfies the differential equation

$$F'(x) + 2xF(x) = 1$$  \hspace{1cm} (6.2)

and has the asymptotic expansion (5, p. 531)

$$F(x) \sim \sum_{s=0}^{\infty} a_s x^{-2s-1},$$  \hspace{1cm} (6.3)

where $a_s = 1 \cdot 3 \cdot 5 \cdots (2s - 1)/(2s+1)$.

There is another approach, however, based on equation (2.11) which is both simpler and more illuminating. Let us begin then by restating the problem in the form

$$L_n \psi_{n}^{(1)} = c_n,$$  \hspace{1cm} (6.4)

where

$$L_n = \frac{d^2}{dx^2} + (2n + \frac{5}{2} - \frac{1}{4}x^2) \frac{d}{dx} + \frac{1}{2}x$$  \hspace{1cm} (6.5)

and $c_n$ are (nonzero) constants the values of which will emerge as part of the solution. For purposes of this discussion it will be convenient to let

$$d_n = \frac{4}{3} c_n.$$  \hspace{1cm} (6.6)
The key to this approach is the observation that

\[ L_n \{ u_n(x) F(\frac{1}{2} x) \} = \frac{3}{2} u_n'' - \frac{3}{4} x u_n' + \left(n + \frac{3}{4}\right) u_n, \]  

(6.7)

where the \( u_n(x) \) are the polynomials discussed in the previous section. Note that the right-hand side of this equation does not involve Dawson's integral but is in fact an (even) polynomial of degree \( 2n \). For \( n = 0 \) we immediately have

\[ \psi_0^{(1)}(x) = F(\frac{1}{2} x), \quad d_0 = 1. \]  

(6.8)

For \( n \geq 1 \) the solutions must be of the form

\[ \psi_n^{(1)}(x) = u_n(x) F(\frac{1}{2} x) - v_n(x), \]  

(6.9)

where \( v_n(x) \) is an (odd) polynomial of degree \( 2n - 1 \). The easiest way of determining \( v_n(x) \) is to recognize that it must be equal to the polynomial part in the asymptotic expansion of \( u_n(x) F(\frac{1}{2} x) \) and \( d_n \) is then the coefficient of \( x^{-1} \) in that expansion. The calculations involved in this procedure are really very simple and, for example, we find that

\[ v_1(x) = x, \quad d_1 = 5; \quad v_2(x) = x^3 + 4x, \quad d_2 = 35. \]  

(6.10)

Note that the normalization of these eigenfunctions is effectively fixed through the normalization of the polynomials \( u_n(x) \).

To illustrate the behaviour of the odd eigenfunctions consider \( \psi_0^{(1)}(x) = F(\frac{1}{2} x) \). In terms of the physical variables we have

\[ \phi_0^{(1)}(z) = 2^{\frac{1}{2}}(\alpha R)^{-\frac{1}{4}} e^{\frac{1}{2} \pi i} F(2^{-\frac{1}{4}} (\alpha R)^{\frac{1}{4}} e^{-\frac{1}{4} \pi i} z), \]  

(6.11)

where \( \phi_0^{(1)}(z) \) has been normalized so that \( \phi_0^{(1)'}(0) = 1 \). The real and imaginary

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parts of $\phi^{(1)}_0(z)$ are shown in Fig. 2(a) for $\alpha R = 10000$ and the corresponding numerical solution is shown in Fig. 2(b). It will be seen in Fig. 2(a) that $\phi^{(0)}_1(z)$ does not satisfy the boundary conditions at $z = 1$ due to its slow algebraic decay and this illustrates the limitations of the second assumption made in section 2 especially for the odd modes.

7. The unbounded odd solutions $\psi^{(3)}_n(x)$

Although the solutions of this type are not needed for the physical problem they are of sufficient intrinsic interest to warrant some discussion. By way of motivation consider the case when $n = 0$ for which the required solution can be found by what is essentially the method of reduction of order. Thus, on setting $n = 0$ in equation (5.4) and then letting

$$u'(x) = v(x)e^{x^2/2}$$  \hspace{1cm} (7.1)

we find that

$$v'' - xv' + \frac{3}{2}v = 0.$$  \hspace{1cm} (7.2)

This is a form of Hermite’s equation and has the solutions

$$He_3(2^{-\frac{1}{2}}x) = 2^{-\frac{3}{4}}(x^3 - 6x)$$  \hspace{1cm} (7.3)

and

$$he_3(2^{-\frac{1}{2}}x) = \frac{1}{2}\left((x^3 - 6x)F'(\frac{1}{2}x) - (x^2 - 4)\right)e^{x^2/2},$$  \hspace{1cm} (7.4)
where \( h_{e_3}(2^{-\frac{1}{2}}x) \) is a Hermite function of the second kind (7). The solution (7.3) simply leads to a multiple of \( \psi_0^{(2)}(x) \). The solution (7.4), however, after integration and renormalization, leads to the desired result

\[
\psi_0^{(3)}(x) = \{(x^2 - 10) F(\frac{1}{2} x) + 8\sqrt{2} F(2^{-\frac{1}{2}} x) - x\} e^{\frac{1}{4} x^2}.
\] (7.5)

There are two aspects of this result worth noting. First, the coefficient of \( F(\frac{1}{2} x) \) is simply \( \psi_0^{(2)}(x) \). Second, when equation (6.3) is used to derive the asymptotic expansion of \( \psi_0^{(3)}(x) \) there is an almost miraculous cancellation of terms and we find that

\[
\psi_0^{(3)}(x) \sim 24x^{-5} e^{\frac{1}{4} x^2} \quad \text{as} \quad x \rightarrow \pm \infty
\] (7.6)

which is in agreement with equation (2.12d).

For \( n \geq 1 \) it is natural to begin by letting

\[
\psi(x) = \hat{u}(x) e^{\frac{1}{4} x^2}.
\] (7.7)

Equation (5.2) then shows that \( \hat{u}(x) \) satisfies the equation

\[
M_n \hat{u}(x) = \hat{u}'' + \frac{3}{2} x \hat{u}'' + (2n + 4 + \frac{1}{2} x^2) \hat{u}' + (n + \frac{5}{2}) x \hat{u} = 0.
\] (7.8)

Note that this equation admits solutions which behave like \( x^{-2n-5} \) as \( x \rightarrow \pm \infty \).

Although it is possible to derive integral representations for the solutions of equation (7.8), they are not of a particularly transparent sort and instead we will proceed in a manner similar to the one discussed in the previous section. For that purpose we observe that

\[
M_n \{ \psi_n^{(2)}(x) F(\frac{1}{2} x) \} = (n + 3) x^2 - 2(4n^2 + 11n + 6)
\] (7.9)

and

\[
M_n \{ u_n(x) \sqrt{2} F(2^{-\frac{1}{2}} x) \} = 3u_n'' + 2(n + 1) u_n.
\] (7.10)
Note that the right-hand sides of these equations do not involve Dawson’s integral.

We also observe that

\[ M_n \{x^{2p+1}\} = (n + p + 3)x^{2p+2} + (2p + 1)(2n + 3p + 4)x^{2p} + 2p(4p^2 - 1)x^{2p-2} \quad (p = 0, 1, 2, \ldots). \]  \hspace{1cm} (7.11)

These results then suggest that \( \hat{u}_n(x) \) for \( n \geq 1 \) must be of the form

\[ \hat{u}_n(x) = \psi_n^{(2)}(x) \ F(\frac{1}{2}x) + \hat{c}_n u_n(x) \sqrt{2} \ F(2^{-\frac{1}{2}}x) - \hat{v}_n(x), \]  \hspace{1cm} (7.12)

where the \( \hat{c}_n \) are constants to be determined and the \( \hat{v}_n(x) \) are (odd) polynomials of degree \( 2n - 1 \).

Let us illustrate this procedure for \( n = 1 \). In that case we have

\[ \hat{u}_1(x) = \psi_1^{(2)}(x) \ F(\frac{1}{2}x) + \hat{c}_1 u_1(x) \sqrt{2} \ F(2^{-\frac{1}{2}}x) - \hat{a}_1 x. \]  \hspace{1cm} (7.13)

On applying the operator \( M_1 \) to this equation and then using equations (7.9) to (7.11) we see that

\[ M_1 \{\hat{u}_1(x)\} = 4(1 - \hat{a}_1 + \hat{c}_1)x^2 - 6(7 + \hat{a}_1 - 3\hat{c}_1). \]  \hspace{1cm} (7.14)

Thus, if \( \hat{a}_1 = 5 \) and \( \hat{c}_1 = 4 \) then we have the solution

\[ \psi_1^{(3)}(x) = \{ (x^2 - 18) \ F(\frac{1}{2}x) + 4(x^2 + 3)\sqrt{2} \ F(2^{-\frac{1}{2}}x) - 5x \} e^{\frac{1}{4}x^2}, \]  \hspace{1cm} (7.15)

where \( \psi_1^{(3)}(x) \sim 120x^{-7} e^{\frac{1}{4}x^2} \) as \( x \to \pm \infty \). In a similar way we find that

\[ \psi_2^{(3)}(x) = \{ (x^2 - 26) \ F(\frac{1}{2}x) + (x^4 + 2x^2 + 19) \ F(2^{-\frac{1}{2}}x) - (x^3 + 4x) \} e^{\frac{1}{4}x^2}, \]  \hspace{1cm} (7.16)

where \( \psi_2^{(3)}(x) \sim 840x^{-9} e^{\frac{1}{4}x^2} \).
8. Discussion

The problem considered in this paper clearly has a significance which transcends its hydrodynamical origins. Equation (2.8), for example, can be regarded as a fourth-order generalization of Weber’s equation for the parabolic cylinder functions and, from that point of view, a whole host of directions for further study present themselves. One such direction would be the derivation of integral representations of the solutions for $\nu \in \mathbb{C}$. That is a formidable task, well beyond the scope of this paper, but it would presumably resolve the enumeration problem for the eigenvalues of the odd modes and show that $\mu = 2n + \frac{5}{2}$ ($n = 0, 1, 2, \ldots$) are in fact the exact eigenvalues for both the even and odd modes.

There is, however, one other aspect of the problem worth mentioning and that is concerned with the approximation of $\psi_n^{(0)}(x)$ and $\psi_n^{(1)}(x)$ when $\mu = 2n + \frac{5}{2}$ is large. Approximations of this type can easily be obtained from the uniform approximations given in sections 3 and 4, and they are analogous to the well-known approximation for Hermite polynomials of large degree in terms of Airy functions (5, p. 403).

For the even eigenfunctions we first rewrite equation (5.1) in the form

$$\psi_n^{(0)}(2\sqrt{\mu} t) = u_n(2\sqrt{\mu} t) e^{-\mu t^2} \quad (8.1 \text{a})$$

$$\sim 2^{2n} \mu^n t^{2n} e^{-\mu t^2} \quad \text{as} \quad t \to +\infty. \quad (8.1 \text{b})$$

The uniform approximation to $\psi_n^{(0)}(2\sqrt{\mu} t)$ for $\mu$ large is then given by

$$\psi_n^{(0)}(2\sqrt{\mu} t) \sim C_n \{A(t) \text{Ai}(\zeta, 2) + e^2 B(t) \text{Ai}(\zeta, 1) + e C(t) \text{Ai}(\zeta, 0)\}. \quad (8.2)$$

To determine the constants $C_n$ we first let $t \to \infty$ in the outer expansion of (8.2)
and then match the result to (8.1 b). In this way we find that

\[ C_n = 2^{-\frac{3}{2}} \pi^{\frac{1}{2}} (\mu + \frac{3}{4})^\frac{\mu - \frac{4}{3} \mu - 1 + O(\mu^{-2})}. \]  

(8.3)

For the odd eigenfunctions we have

\[ \psi_n^{(1)}(x) \sim d_n x^{-1}, \]  

(8.4)

where \( d_0 = 1, \ d_1 = 5, \ d_2 = 35, \ldots \) and this can be rewritten in the form

\[ \psi_n^{(1)}(2\sqrt{\mu} t) \sim \frac{1}{2} d_n \mu^{-\frac{1}{2}} t^{-1}. \]  

(8.5)

The uniform approximation to \( \psi_n^{(1)}(2\sqrt{\mu} t) \) is then given by

\[ \psi_n^{(1)}(2\sqrt{\mu} t) \sim D_n \{ \epsilon^{-1} G(t) + A(t) \text{Gi}(\zeta, 2) + \epsilon^2 B(t) \text{Gi}(\zeta, 1) + \epsilon C(t) \text{Gi}(\zeta, 0) \} \]  

(8.6 a)

\[ \sim \frac{3}{2} D_n (ct)^{-1} \quad \text{as} \quad t \to +\infty. \]  

(8.6 b)

Thus, matching (8.6 b) to (8.5) gives

\[ D_n = \frac{3}{4} \epsilon \mu^{-\frac{1}{2}} d_n. \]  

(8.7)

These approximations to \( \psi_n^{(0)}(2\sqrt{\mu} t) \) and \( \psi_n^{(1)}(2\sqrt{\mu} t) \) were derived, of course, on the assumption that \( \mu \) is large. Numerical evaluation for \( n = 0 \) and \( 1 \), however, lead to results which are indistinguishable from the corresponding numerical solutions except near \( t = 0 \).

Despite the intrinsic interest of these approximations, they are not well suited to the calculation of the physical eigenfunctions \( \phi_{n}^{(0)}(z) \) and \( \phi_{n}^{(1)}(z) \) for which evaluation along the ray \( \text{ph} \ x = -\frac{1}{3} \pi \) is required. For that purpose we must use the exact solutions (3.8) and (6.9). To illustrate the behaviour of the higher modes let us
consider the case when \( n = 5 \). In that case we have

\[
\begin{align*}
\psi_5(x) &= x^{10} - 25x^8 + 310x^6 - 630x^4 + 5205x^2 + 16755, \\
v_5(x) &= x^9 - 23x^7 + 272x^5 - 190x^3 + 6345x,
\end{align*}
\]

and \( d_5 = 45045 \). If we now replace \( x \) by \( (2\lambda)^{\frac{1}{2}} e^{-\frac{1}{2} \pi i} z \) in (3.8) and (6.9) and fix the normalization by requiring that \( \phi_5^{(0)}(0) = 1 \) and \( \phi_5^{(1)}(0) = 1 \) then we obtain the results shown in Fig. 3. A striking feature of these results is that they are in the form of wavepackets and that raises the interesting question of whether this behaviour is related to the ideas discussed by Tatsumi and Gotoh (10). They showed quite generally that the eigenfunction for a mode with finite damping at large Reynolds numbers has the form of a concentrated wavepacket. Their analysis differs significantly from ours and further study would be required to establish the precise relationship, if any, between the two approaches. In any event, the eigenfunctions shown in Fig. 3 also show an uncanny resemblance to the numerical results obtained by Davey (11) for the Blasius boundary layer.
REFERENCES


APPENDIX

The generalized Airy and Scorer functions

In this Appendix we wish to record some properties of the generalized Airy and Scorer functions which were needed in sections 3 and 4. The particular generalization needed here was motivated in part by the discussion given in (8). Some of the asymptotic expansions given below can be found in (9) and they can all be derived from the results given in the Appendix of (6).

The Airy functions $\text{Ai}(x)$ and $\text{Bi}(x)$ are, of course, solutions of Airy’s equation $w'' - xw = 0$. The required generalization of $\text{Ai}(x)$ is then obtained by letting

$$\text{Ai}(x, 0) = \text{Ai}(x) \quad \text{and} \quad \text{Ai}(x, p) = -\int_x^\infty \text{Ai}(t, p - 1) \, dt,$$

(A.1)

where $p$ is an integer. In the present paper we need two terms in the asymptotic expansions of $\text{Ai} \ (\pm x, 2)$ as $x \to +\infty$. If we let $\xi = \frac{2}{3} x^{\frac{3}{2}}$ as usual then we have

$$\text{Ai}(x, 2) \sim \frac{1}{2} \pi^{-\frac{1}{2}} \ x^{-\frac{1}{2}} \ e^{-\xi} \ (1 - \frac{101}{72} \xi^{-1} + \cdots)$$

(A.2)

and

$$\text{Ai}(-x, 2) \sim x + \pi^{-\frac{1}{2}} \ x^{-\frac{1}{2}} \{\sin(\xi - \frac{3}{4}\pi) - \frac{101}{72} \xi^{-1} \cos(\xi - \frac{3}{4}\pi) + \cdots\}. \quad \text{(A.3)}$$

The corresponding expansions for $p < 2$ can be obtained by differentiation.

In the case of $\text{Bi}(x)$ we let

$$\text{Bi}(x, 0) = \text{Bi}(x) \quad \text{and} \quad \text{Bi}(x, p) = \int_0^x \text{Bi}(t, p - 1) \, dt.$$

(A.4)

For the present purposes we need only the asymptotic expansion of $\text{Bi}(-x, 2)$ as
\[ x \to +\infty \] and it is given by

\[
\text{Bi}(-x, 2) \sim (2\pi)^{-1} 3^{\frac{5}{3}} \Gamma\left(\frac{2}{3}\right) + \\
+ \pi^{-\frac{1}{2}} x^{-\frac{5}{4}} \{ \cos(\xi - \frac{3}{4}\pi) + \frac{101}{72} \xi^{-1} \sin(\xi - \frac{3}{4}\pi) - \cdots \}.
\]  
(A.5)

The Scorer functions \( \text{Gi}(x) \) and \( \text{Hi}(x) \) are solutions of the inhomogeneous equations \( w'' - xw = -\pi^{-1} \) and \( w'' - xw = \pi^{-1} \) respectively. A distinctive feature of these functions is that as \( x \to +\infty \) they behave like

\[
\text{Gi}(x) \sim \frac{1}{\pi x} \quad \text{and} \quad \text{Hi}(-x) \sim \frac{1}{\pi x}.
\]  
(A.6)

They also satisfy the connection formula (5, p. 431)

\[
\text{Gi}(x) + \text{Hi}(x) = \text{Bi}(x).
\]  
(A.7)

Now let

\[
\text{Gi}(x, 0) = \pi \text{Gi}(x) \quad \text{and} \quad \text{Gi}(x, p) = \int_0^x \text{Gi}(t, p - 1) \, dt
\]  
(A.8)

with similar definitions for \( \text{Hi}(x, p) \). For the present purposes we need only the asymptotic expansions of \( \text{Gi}(x, 2) \) and \( \text{Hi}(-x, 2) \) as \( x \to +\infty \) and they are given by

\[
\text{Gi}(x, 2) = x \ln x + \left\{ \frac{1}{3} (\ln 3 + 2\gamma) - 1 \right\} x + \frac{1}{2} 3^{-\frac{1}{3}} \Gamma\left(\frac{2}{3}\right) + O(x^{-2})
\]  
(A.9)

and

\[
\text{Hi}(-x, 2) = x \ln x + \left\{ \frac{1}{3} (\ln 3 + 2\gamma) - 1 \right\} x + 3^{-\frac{1}{3}} \Gamma\left(\frac{2}{3}\right) + O(x^{-2}).
\]  
(A.10)

Note that the constant terms in (A.9) and (A.10) are not the same. From equation (A.7) we also have the connection formula

\[
\text{Gi}(x, p) + \text{Hi}(x, p) = \pi \text{Bi}(x, p).
\]  
(A.11)
For completeness, we also record the recursion formulas satisfied by these generalized Airy and Scorer functions:

\[ \text{Ai}(x, 2) - x\text{Ai}(x, 1) + \text{Ai}(x, -1) = 0, \] (A.12)

\[ \text{Bi}(x, 2) - x\text{Bi}(x, 1) + \text{Bi}(x, -1) = (2\pi)^{-1} 3^{3} \Gamma(\frac{3}{4}), \] (A.13)

\[ \text{Gi}(x, 2) - x\text{Gi}(x, 1) + \text{Gi}(x, -1) = \frac{1}{2} 3^{-\frac{1}{3}} \Gamma(\frac{3}{4}) - x, \] (A.14)

\[ \text{Hi}(x, 2) - x\text{Hi}(x, 1) + \text{Hi}(x, -1) = 3^{-\frac{1}{3}} \Gamma(\frac{3}{4}) + x. \] (A.15)
TABLE 1. Numerical results for the eigenvalues
of the odd modes for $\alpha = 0$ and $\alpha R = 10000$

<table>
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<th>$n$</th>
<th>$4n + 5$</th>
<th>$(1 - c_r)(2\alpha R)^{\frac{1}{2}}$</th>
<th>$-c_i(2\alpha R)^{\frac{1}{2}}$</th>
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</table>
Fig. 1. The behaviour of $\phi_0^{(0)}(z)$ for $\alpha R = 10000$
Fig. 2. (a) The behaviour of $\phi_0^{(1)}(z)$ for $\alpha R = 10,000$.
(b) The corresponding numerical solution.
Fig. 3. The behaviour of $\phi_s^{(0)}(z)$ and $\phi_s^{(1)}(z)$ for $\alpha R = 10000$