Limiting average availability of a system supported by several spares and several repair facilities

Jyotirmoy Sarkar and Fang Li

Department of Mathematical Sciences, Indiana University Purdue University
Indianapolis, 402 N Blackford Street, Indianapolis, IN 46202-3216, USA

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Abstract

A one-unit repairable system, supported by \( r \) repair facilities and \( s \) spare units, fails when all units are down and are undergoing or awaiting repair. Under the perfect repair policy and instantaneous commencement of repair and installation to operation, the limiting average availability of the system is obtained when the lifetime has a density function and the repair time is exponentially distributed. © 2006 Elsevier Science B.V. All rights reserved

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1. Introduction

We consider a one-unit repairable system under continuous monitoring. Initially, one-unit is put on operation and \( s \) identical spares remain on cold standby. As soon as the operating unit fails one of the spare units, if available, is placed on service (this is called instantaneous installation to operation). The failed unit joins the repair queue. There

\(^1\)Corresponding author. Tel.: +1-317-274-8112; fax: +1-317-274-3460; email: jsarkar@math.iupui.edu.
are \( r \) repair facilities. Repair starts on a failed unit as soon as one of the repair facilities becomes free (this is called instantaneous commencement of repair). We assume that repair takes a random amount of time, after which the unit is restored back to a level equivalent to a new unit (this is called the perfect repair policy) and becomes a viable spare. A spare unit cannot fail, hence is said to remain on cold standby until placed on service. Since there are only a finite number of spares, a replacement of the failed operating unit may not be always possible. The system fails when the operating unit fails and no spare unit is available to replace it, because all units are either undergoing or awaiting repair. Specifically, this implies that \( r \leq s + 1 \), since otherwise some repair facilities will always remain idle.

Of interest is the limiting average availability (the limiting expected proportion of time the system is up), which is defined as

\[
A_{av} := \frac{\text{MSUT}}{\text{MSUT} + \text{MSDT}},
\]

(1.1)

where MSDT stands for mean system down time and is the mean duration from the moment the system fails until it is revived through repair, and MSUT stands for mean system up time and is the mean duration from the epoch when the failed system is revived to the next system failure. Barlow and Proschan (1975, page 206) provide the limiting average availability only for the case of one repair facility and either no or one spare unit, assuming exponential lifetime and exponential repair time distributions. However, their expression (3.6) for the limiting average availability of a one-unit system supported by one repair facility and one spare unit, when lifetime and repair time distributions are arbitrary, is incorrect as pointed out by Sen and Bhattacharjee (1986, page 283) who derived the correct expression.

A more ambitious objective is to obtain an expression for the instantaneous availability function \( A(t) \), the probability that the system is up at a specified time \( t > 0 \). \( A(t) \) measures the performance of a maintained system and is an important aspect of reliability theory. For an excellent account on the subject see, for example, Høyland and Rausand (1994). Explicit evaluation of exact availability is often very difficult. For some examples see Sarkar and Chaudhuri (1999), Biswas and Sarkar (2000) and Sarkar and Sarkar (2000, 2001) and Biswas et. al. (2003). No attempt is made in this paper to
evaluate $A(t)$ in our model.

In many practical situations, the limiting average availability is sufficient for decision making. In this paper, therefore, we focus on the limiting average availability of a one-unit system supported by $r$ repair facilities and $s$ spares. We assume all lifetimes are independent and identically distributed (IID) with an absolutely continuous cumulative distribution function (CDF) (with density function $f$), while all repair times are IID exponential($\beta$). Furthermore, the lifetimes and the repair times are stochastically independent. Admittedly, the assumption of exponential repair times is a limitation that we are unable to circumvent, since our method exploits the lack of memory property of the exponential distribution. Nonetheless, the generality of the lifetime CDF renders our result applicable to many practical instances.

For the exponential($\beta$) repair time model, MSDT is simply $(r\beta)^{-1}$. This is because the earliest repair time, among those of the $r$ units that are under repair when the system is down, follows an exponential($r\beta$) distribution. Obtaining the expression for MSUT is a lot more complicated even for the case when both lifetime and repair time are exponentially distributed. This precisely is the main contribution of this article. We find the expression for MSUT by solving a system of linear equations that are set up by utilizing the lack of memory property of the exponential distribution. We first consider exponential lifetime in Section 2. Since both lifetime and repair time distributions are exponential, we are able to provide an explicit algebraic solution to the system of linear equations. Next, in Section 3, we derive the expression for MSUT for continuous lifetime distributions. We give an illustrative example in which we numerically compute the limiting average availability for various lifetime distributions. In Section 4 we provide a summary and a discussion of the nature of the difficulty in extending our work to arbitrary lifetime and arbitrary repair time distributions, unless $(r = 1, s = 1)$.

2. MSUT and $A_{av}$ when lifetime is exponential

Assume that the lifetimes of the units are IID with exponential($\alpha$) distribution; and the repair times are IID with exponential($\beta$) distribution. Assume also that the lifetimes are independent of the repair times.
We observe the system only at epochs when either an operating unit fails or a down unit is repaired, and keep track of the number of units that are down at these epochs. We say that the embedded discrete-time Markov chain (tracked at the observation epochs) is in state $i$ ($i = 0, 1, \ldots, s, s + 1$) if there are $i$ failed units undergoing or awaiting repair. Note that the system is in state 0 at time $t = 0$. In state $s + 1$, the system is down, while in all other states the system is up.

From any state $i$ the system moves to an adjacent state after a random sojourn time $\xi_i$. Using the lack of memory property of the exponential distribution, it is straightforward to see that the sojourn time $\xi_i$ in state $i$ is exponentially distributed with scale parameter $a_i \alpha + b_i \beta$, where $a_i = 1$ if $0 \leq i \leq s$ and $a_{s+1} = 0$, and $b_i = \min\{i, r\}$. Furthermore, the transition probabilities $q_{ij}$ from state $i$ to state $j$ ($i, j = 0, 1, \ldots, s + 1$) are given by

$$
q_{i,i+1} = \frac{a_i \alpha}{a_i \alpha + b_i \beta} \\
q_{i,i-1} = \frac{b_i \beta}{a_i \alpha + b_i \beta} \\
q_{i,j} = 0 \text{ for } |i-j| \neq 1.
$$

Note from (2.1) that states 0 and $s + 1$ are reflecting states (that is, they move to states 1 and $s$ respectively, with probability one). Also, note that the Markov chain is irreducible (that is, all states form a single recurrent class).

When the system is down, all $r (\leq s + 1)$ repair facilities are busy, each repairing a failed unit. Therefore, as mentioned in Section 1, the system down time, being the minimum of $r$ independent exponential($\beta$) random variables, has an exponential($r \beta$) distribution. Hence,

$$
\text{MSDT} = (r \beta)^{-1}.
$$

In order to find the expression for MSUT, let $E_i$ denote the mean time to move from state $i$ to state $i + 1$ for $i = 0, 1, \ldots, s$. Note that the failed system (in state $s + 1$), when revived, enters state $s$ in which only one-unit is operating and all other units are undergoing or awaiting repair (hence, there is no viable spare). The revived system remains operational until the next system failure, that is, until it enters state $s + 1$. Therefore, MSUT is the same as $E_s$. 

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How do we find an expression for $E_s$? We develop a system of equations relating $E_0, E_1, \ldots, E_s$. Note that $E_0 = 1/\alpha$ is simply the mean lifetime of a unit, and for $i = 1, \ldots, s$, we have

$$E_i = \frac{1}{a_i \alpha + b_i \beta} + \frac{b_i \beta}{a_i \alpha + b_i \beta} (E_{i-1} + E_i). \tag{2.3}$$

The first term on the right hand side of (2.3) is the expected sojourn time in state $i$, and the second term is the product of $q_{i,i-1}$ and the additional mean time to move from state $i - 1$ to state $i + 1$ (of course, via state $i$). Note also that if the system moves directly from state $i$ to state $i + 1$, no additional time is required to reach state $i + 1$. Hence, $q_{i,i+1}$ is multiplied by 0, and therefore, is absent from the right hand side of (2.3). After simplification, (2.3) yields

$$a_i \alpha E_i = 1 + b_i \beta E_{i-1}.$$ 

Hence, substituting the values of $a_i$ and $b_i$, and letting $\rho = \beta/\alpha$, we have

$$E_0 = 1/\alpha, \quad E_i - (i \wedge r) \rho \ E_{i-1} = 1/\alpha, \quad \text{for } i = 1, \ldots, s. \tag{2.4}$$

Theorem 2.1 below gives an expression for $E_s$ by solving the system of linear equations (2.4).

**Theorem 2.1.** For a one-unit system supported by $r \geq 1$ repair facilities and $s \geq r - 1$ spare units, with lifetime distribution exponential($\alpha$) and repair time distribution exponential($\beta$), the (long-run) mean system up time is given by

$$MSUT = E_s = \alpha^{-1} \sum_{i=0}^{s} \gamma_i \rho^{s-i}, \tag{2.5}$$

where $\rho = \beta/\alpha$ and

$$\gamma_i = \begin{cases} r^{s-r} \ r! / i! & \text{if } i = 0, 1, \ldots, r - 1 \\ r^{s-i} & \text{if } i = r, \ldots, s. \tag{2.6} \end{cases}$$

Hence, the limiting average availability is given by

$$A_{av} = \frac{r \rho \sum_{i=0}^{s} \gamma_i \rho^{s-i}}{1 + r \rho \sum_{i=0}^{s} \gamma_i \rho^{s-i}}. \tag{2.7}$$

**Proof.** Rewriting (2.4) in matrix notation we have
\[
\begin{pmatrix}
1 & 0 & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\
-\rho & 1 & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\
0 & -2\rho & 1 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & 0 & -r\rho & 1 & \ldots & \ldots & 0 \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \vdots \\
0 & \ldots & \ldots & 0 & -r\rho & 1 & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & \ldots & 0 & 0 & -r\rho \\
0 & \ldots & \ldots & \ldots & 0 & \ldots & \ldots & 0 \\
\end{pmatrix}
= \begin{pmatrix}
E_0 \\
E_1 \\
E_2 \\
E_r \\
E_{r+1} \\
E_s \\
\end{pmatrix} = \frac{1}{\alpha} \begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
\end{pmatrix}.
\tag{2.8}
\]

Denoting the matrix on the left hand side of (2.8) by \( D \), we have
\[
E_s = \frac{1}{\alpha} \begin{pmatrix}
(0, 0, \ldots, 0, 1) \\
(1, 1, \ldots, 1, 1) \\
\end{pmatrix}^T = \frac{1}{\alpha} \sum_{i=0}^{s} d^{si}.
\tag{2.9}
\]

Thus, \( E_s \) equals the mean lifetime of a unit multiplied by the sum of the elements in the last row of \( D^{-1} = ((d^{ij})) \) where \( 0 \leq i, j \leq s \). From linear algebra, we know that
\[
d^{si} = (-1)^{s+i} \det(D_{is})/ \det(D),
\tag{2.10}
\]
where \( D_{is} \) is the \((i, s)\)-th cofactor of \( D \). Note that \( \det(D) = 1 \) and
\[
\det(D_{is}) = \begin{cases}
(-(i+1)\rho) \cdots [-r\rho] \times [-r\rho]^{s-r} & \text{if } i = 0, 1, \ldots, r - 1 \\
[-r\rho]^{s-i} & \text{if } i = r, \ldots, s \\
\gamma_i (-\rho)^{s-i}.
\end{cases}
\]

Therefore, (2.10) simplifies to \( d^{si} = \gamma_i \rho^{s-i} \); and hence, (2.9) simplifies to (2.5). Finally, (2.7) follows from (1.1), (2.2) and (2.5).

The special cases when \( r = 1 \) and \( r = 2 \) are given by the following Corollary.

**Corollary 2.2.** For a one-unit system supported by \( r = 1 \) repair facility and \( s \geq 0 \) spare units, with lifetime distribution exponential(\( \alpha \)) and repair time distribution exponential(\( \beta \)), we have \( \gamma_i = 1 \) for \( i = 0, \ldots, s \). Therefore,
\[
E_s(r = 1) = \begin{cases}
\frac{1}{\alpha} \frac{e^{s+1} - 1}{\rho - 1} & \text{if } \rho \neq 1 \\
\frac{1}{\alpha} (s + 1) & \text{if } \rho = 1,
\end{cases}
\tag{2.11}
\]

where \( \rho = \beta / \alpha \), and

\[
A_{av}(r = 1) = \begin{cases} 
\frac{\rho^{s+1} - 1}{\rho^{s+2} - 1} = \frac{\beta^{s+1} - \alpha^{s+1}}{\beta^{s+2} - \alpha^{s+2}} & \text{if } \rho \neq 1 \\
\frac{s+1}{s+2} & \text{if } \rho = 1.
\end{cases}
\] (2.12)

Likewise, for a one-unit system supported by \( r = 2 \) repair facilities and \( s \geq 1 \) spare units, with lifetime distribution exponential(\( \alpha \)) and repair time distribution exponential(\( \beta \)), we have \( \gamma_0 = 2^{s-1} \) and \( \gamma_i = 2^{s-i} \) for \( i = 1, \ldots, s \). Therefore,

\[
E_s(r = 2) = \begin{cases} 
\frac{1}{2^s} \left( \frac{(2\rho)^{s+1} + (2\rho)^s - 2}{2\rho - 1} \right) & \text{if } \rho \neq 1/2 \\
\frac{1}{2^s} (2s + 1) & \text{if } \rho = 1/2
\end{cases}
\] (2.13)

and

\[
A_{av}(r = 2) = \begin{cases} 
\frac{\rho[(2\rho)^{s+1} + (2\rho)^s - 2]}{2s+1} & \text{if } \rho \neq 1/2 \\
\frac{2s+1}{2s+3} & \text{if } \rho = 1/2.
\end{cases}
\] (2.14)

**Example 2.1.** Suppose that the mean lifetime of each unit is 50 days and the mean repair time is exponentially distributed with mean 35 days. At present there is only one repair facility and two spare units supporting the system. In order to improve the system availability should we (a) buy one more spare unit, or (b) set up another repair facility? Assume the cost is about the same for each alternative.

Assuming further that the lifetime of each operating unit is exponential(1/50), we have \( \rho = 10/7 \). From (2.2), we have MSDT(\( r = 1 \)) = 35 and MSDT(\( r = 2 \)) = 17.5. From (2.11), we have \( E_s(r = 1, s) = 50[(10/7)^{s+1} - 1]/(3/7) \); so that \( E_2(r = 1, s = 2) = 223.5 \) and \( E_3(r = 1, s = 3) = 369.2 \). Also from (2.13), we have \( E_s(r = 2) = 25[(20/7)^{s+1} + (20/7)^{s-2}]/(13/7) \); so that \( E_2(r = 2, s = 2) = 396.9 \). Therefore, the current limiting average availability is \( A_{av}(r = 1, s = 2) = .8646 \). If we follow option (a), availability will increase to \( A_{av}(r = 1, s = 3) = .9134 \). On the other hand, if we follow option (b), availability will increase to \( A_{av}(r = 2, s = 2) = .9578 \). Therefore, option (b) is preferable to option (a).
3. MSUT and $A_{av}$ when lifetime has an arbitrary continuous distribution

In this section we extend the results of the previous section by assuming that lifetimes are IID having absolutely continuous CDF $F$ with density function $f$ and mean $\mu$. Note that since the support of $f$ is the positive half line, the Laplace transformation $\int_0^\infty e^{-\beta x} f(x) \, dx$ of $f$ exists. We continue to assume that the repair times are IID having exponential($\beta$) distribution and that the lifetimes are independent of the repair times.

Since we no longer assume the system lifetime distribution to have the lack of memory property, we modify the observation times in the current set up. Now we observe the system only at epochs when an operating unit, if any, fails or when a down system is revived, and keep track of the number of units that are down at each newly defined observation time. We say that the embedded Markov chain (tracked at the modified observation times) is in state $i$ ($i = 0, 1, \ldots, s, s + 1$) if there are $i$ failed units undergoing or awaiting repair. Note that at time $t = 0$, the system is in state 0 and it never returns to state 0. Thus, state 0 is transient. It will be evident from the transition probabilities described below that the remaining states $\{1, 2, \ldots, s + 1\}$ form a single recurrent class.

When the system enters state $s + 1$, it fails; and it can only move to state $s$ at the next observation time (when the system is revived) after a random amount of time having exponential($r\beta$) distribution. Hence, MSDT = $(r\beta)^{-1}$ as in Section 2.

In the sequel, we consider any particular state $i$ ($1 \leq i \leq s$). If the system is in state $i$ at an observation time, the duration until the next observation time is the same as the lifetime $X$ of the operating unit, and so it has a density function $f$. Let $N_i$ denote the number of repairs completed during the lifetime of the current operating unit. Note that $N_i$ takes values in $\{0, 1, \ldots, i\}$. Then starting from state $i$ at one observation time the system enters state $j$ at the next observation time with transition probabilities given by

$$q_{i,j} = P\{N_i = i + 1 - j\}, \quad \text{for } 1 \leq j \leq i + 1. \quad (3.1)$$

The probability mass function of $N_i$ is presented in Lemma 3.1 below. Towards that end, let $Y_{k|i}$ ($1 \leq k \leq i$) denote the time when $k$ out of the $i$ failed units are repaired. What is the density function $f_{Y_{k|i}}(y)$ of $Y_{k|i}$?
For $1 \leq i \leq r$, when the system enters state $i$, all $i$ units are under repair. By the lack of memory property of the exponential distribution, the successive differences $Y_{1i}, \ldots, (Y_{k+1|i} - Y_{k|i}), \ldots, (Y_{i|i} - Y_{i-1|i})$ have independent exponential distributions with parameters $i\beta, \ldots, (i-k)\beta, \ldots, \beta$, respectively. Equivalently, $Y_{ki}$ is the $k$-th order statistic among $i$ IID exponential($\beta$) random variables.

However, for $r + 1 \leq i \leq s$, when the system enters state $i$ there are $r$ failed units undergoing repair and $i - r$ failed units awaiting repair. As soon as a repair facility finishes repair on one-unit, it starts repairing one of the waiting units. Again by the lack of memory property of the exponential distribution, the successive differences $Y_{1i}, (Y_{2i} - Y_{1i}), \ldots, (Y_{i-r+1|i} - Y_{i-r|i}), (Y_{i-r+2|i} - Y_{i-r+1|i}), \ldots, (Y_{ji} - Y_{i-1|i})$ have independent exponential distributions with parameters $r\beta, r\beta, \ldots, r\beta, (r-1)\beta, \ldots, \beta$, respectively. Therefore, for $r + 1 \leq i \leq s$, if $1 \leq k \leq i - r$, $Y_{ki}$ is a gamma($k, r\beta$) random variable; whereas if $i - r + 1 \leq k \leq i$, the distribution of $Y_{ki}$ is the same as that of the sum of two independent random variables, one of which is a gamma($i-r, r\beta$) random variable and the other is the $(k - i + r)$-th order statistics among $r$ IID exponential($\beta$) random variables.

Hence, in summary, the density function $f_{Y_{ki}}(y)$, $y \geq 0$ of $Y_{ki}$ is given by

$$
\begin{align*}
  f_{Y_{ki}}(y) &= k \binom{i}{k} (1 - e^{-\beta y})^{k-1} \beta e^{-\beta y} e^{-(i-k)\beta y}, & \text{for } 1 \leq i \leq r \text{ and } 1 \leq k \leq i, & (3.2) \\
  &= \frac{(r\beta)^k}{\Gamma(k)} y^{k-1} e^{-r\beta y}, & \text{for } r + 1 \leq i \leq s \text{ and } 1 \leq k \leq i - r, & (3.3) \\
  &= \frac{(r\beta)^{i-r+1}}{\Gamma(i-r)} \binom{i}{k-i+r-1} e^{-\beta y} e^{-(i-k+1)\beta y} \int_0^y e^{i-r-1} \left(e^{-\beta z} - e^{-\beta y}\right)^{k-i+r-1} dz, & \text{for } r + 1 \leq i \leq s \text{ and } i - r + 1 \leq k \leq i, & (3.4)
\end{align*}
$$

where $\binom{0}{0}$ is defined to be $1$.

**Lemma 3.1.** The probability mass function of $N_i$ is given by

$$
P\{N_i = 0\} = \int_0^{\infty} e^{-(i+r)\beta x} f(x) \, dx
$$

and for $1 \leq k \leq i$, $P\{N_i = k\}$ is given by

$$
\binom{i}{k} \int_0^{\infty} (1 - e^{-\beta x})^k e^{-(i-k)\beta x} f(x) \, dx, \quad \text{for } 1 \leq i \leq r \text{ and } 1 \leq k \leq i,
$$

(3.6)
\[
\frac{(r\beta)^k}{\Gamma(k+1)} \int_0^\infty x^k e^{-r\beta x} f(x) \, dx, \quad \text{for } r + 1 \leq i \leq s \text{ and } 1 \leq k \leq i - r, \tag{3.7}
\]
\[
\frac{(r\beta)^{i-r+1}}{\Gamma(i-r)} \left( \frac{r-1}{k-i+r-1} \right) \times 
\int_0^\infty \int_0^x \int_0^y z^{i-r-1} (e^{-\beta z} - e^{-\beta y})^{k-i-r-1} \, dz \, e^{-\beta y} \, dy \, e^{-(i-k)\beta x} f(x) \, dx, \tag{3.8}
\]
for \( r + 1 \leq i \leq s \) and \( i - r + 1 \leq k \leq i \).

**Proof.** For \( 1 \leq i \leq s \), if \( k = 0 \), then we have
\[
P\{N_i = 0\} = P\{X \leq Y_{1|i}\} = \int_0^\infty P\{Y_{1|i} \geq x\} \, f(x) \, dx = \int_0^\infty e^{-(i\wedge r)\beta x} f(x) \, dx
\]
since \( Y_{1|i} \) has exponential((\( i \wedge r \))\( \beta \)) distribution.

Next, for \( 1 \leq i \leq r \), if \( 1 \leq k \leq i - 1 \), then using (3.2), we have
\[
P(N_i = k) = P\{Y_{k|i} \leq X < Y_{k+1|i} + (Y_{k+1|i} - Y_{k|i})\}
= \int_0^\infty \int_0^x P\{Y_{k+1|i} - Y_{k|i} > x - y\} \, f_{Y_{k|i}}(y) \, dy \, f(x) \, dx
= \int_0^\infty \int_0^x k(i/k)(1-e^{-\beta y})^{k-1} \beta e^{-\beta y} \, dy \, e^{-(i-k)\beta x} f(x) \, dx
= \binom{i}{k} \int_0^\infty (1-e^{-\beta x})^k e^{-(i-k)\beta x} f(x) \, dx.\]

Similarly, for \( 1 \leq i \leq r \), if \( k = i \), then we have
\[
P(N_i = i) = P\{Y_{i|i} \leq X\} = \int_0^\infty (1-e^{-\beta x})^i f(x) \, dx.
\]
This completes the proof of (3.6).

Finally, for \( r + 1 \leq i \leq s \), we use densities (3.3) and (3.4) and imitate the same arguments as in establishing (3.6), to prove (3.7) and (3.8). The details are omitted. □

In view of (3.1), Lemma 3.1 provides a recipe for obtaining the transition probabilities \( q_{i,j} \) for \( 1 \leq i \leq s \) and \( 1 \leq j \leq i + 1 \). We next turn attention to the expected transition times. As in Section 2, let \( E_i \) denote the mean time to move from state \( i \) to state \( i + 1 \) for \( i = 1, 2, \ldots, s \). Then
\[
E_i = \mu + q_{i1}(E_1 + \ldots + E_i) + q_{i2}(E_2 + \ldots + E_i) + \ldots
+ q_{i,i-1}(E_{i-1} + E_i) + q_{ii}E_i + q_{i,i+1} \times 0. \tag{3.9}
\]
The first term on the right hand side of (3.9) is the mean duration between successive observation times (which is the same as the mean lifetime \( \mu \) of the operating unit), the second term is the product of \( q_{i,1} \) and the mean time to move from state 1 to state \( i + 1 \) (via states \( 2, \ldots , i \)), the third term is the product of \( q_{i,2} \) and the mean time to move from state 2 to state \( i + 1 \) (via states \( 3, \ldots , i \)), and so on.

Again as in Section 2, we have \( \text{MSDT}= (r\beta)^{-1} \text{ and MSUT} = E_s \). Our objective is to find \( E_s \) by solving the system of linear equations (3.9), which we can rewrite as

\[
E_i = \mu + q_{i,1} E_1 + (q_{i,1} + q_{i,2}) E_2 + \cdots + (q_{i,1} + \cdots + q_{i,i}) E_i
\]

where \( Q_{ij} = \sum_{h=1}^{j} q_{ih} \) for \( 1 \leq j \leq i \). In view of (3.1),

\[
Q_{ij} = P\{i + 1 - j \leq N_i \leq i\} = 1 - P\{0 \leq N_i \leq i - j\}, \quad \text{for } 1 \leq j \leq i \tag{3.11}
\]

is the probability that at least \( i + 1 - j \) out of the \( i \) failed units are repaired during the lifetime of the operating unit. Note that \( Q_{ij} \) can be evaluated using Lemma 3.1.

**Theorem 3.1.** For a system supported by \( r \geq 1 \) repair facilities and \( s \geq (r-1) \lor 1 \) spare units, with lifetime \( X \) having density \( f \) and mean \( \mu \), and with repair times exponential(\( \beta \)), the (long-run) mean system up time is given by

\[
\text{MSUT} = E_s = \mu (0, \cdots , 0, 1)(I - Q)^{-1}(1, \cdots , 1)^T, \tag{3.12}
\]

where \( I \) denotes the \( s \times s \) identity matrix and \( Q \) denotes the \( s \times s \) lower-triangular matrix with entries \( \{Q_{ij}, 1 \leq j \leq i\} \) given in (3.11).

Hence, the limiting average availability is given by

\[
A_{av} = \frac{\mu (0, \cdots , 0, 1)(I - Q)^{-1}(1, \cdots , 1)^T}{\mu (0, \cdots , 0, 1)(I - Q)^{-1}(1, \cdots , 1)^T + (r\beta)^{-1}}. \tag{3.13}
\]

**Proof.** Writing (3.10) in the matrix notation, we have

\[
\begin{pmatrix}
E_1 \\
E_2 \\
\vdots \\
E_{s-1} \\
E_s
\end{pmatrix}
= \mu \begin{pmatrix}
1 \\
1 \\
\vdots \\
1 \\
1
\end{pmatrix}
+ \begin{pmatrix}
Q_{11} & 0 & \cdots & \cdots & 0 \\
Q_{21} & Q_{22} & \ddots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
Q_{s-1,1} & Q_{s-1,2} & \cdots & Q_{s-1,s-1} & 0 \\
Q_{s,1} & Q_{s,2} & \cdots & Q_{s-1,s} & Q_{s,s}
\end{pmatrix}
\begin{pmatrix}
E_1 \\
E_2 \\
\vdots \\
E_{s-1} \\
E_s
\end{pmatrix}. \tag{3.14}
\]
The theorem follows.

To illustrate the computation in Theorem 3.1, let us revisit Example 2.1, with the provision that the lifetime $X$ may have an arbitrary absolutely continuous CDF $F$ with density function $f$. In the example, we calculate MSUT and $A_{av}$ for $(r = 1, s = 2)$, $(r = 1, s = 3)$ and $(r = 2, s = 2)$.

For $(r = 1, s = 2)$, by (3.11) and Lemma 3.1, $Q$ in (3.14) simplifies to

$$
 Q = \begin{pmatrix}
 1 - E[e^{-\beta X}] & 0 \\
 1 - E[e^{-\beta X}(1 + \beta X)] & 1 - E[e^{-\beta X}]
 \end{pmatrix},
$$

where $E$ denotes mathematical expectation. Hence, we have

$$
 E_2(r = 1) = \mu (0, 1)(I - Q)^{-1}(1, 1)^T = \mu \frac{1 - E[\beta X e^{-\beta X}]}{(E[e^{-\beta X}])^2}. \quad (3.15)
$$

For $(r = 1, s = 3)$, $Q$ simplifies to

$$
 Q = \begin{pmatrix}
 1 - E[e^{-\beta X}] & 0 & 0 \\
 1 - E[e^{-\beta X}(1 + \beta X)] & 1 - E[e^{-\beta X}] & 0 \\
 1 - E[e^{-\beta X}(1 + \beta X + \frac{(\beta X)^2}{2})] & 1 - E[e^{-\beta X}(1 + \beta X)] & 1 - E[e^{-\beta X}]
 \end{pmatrix}.
$$

Hence, we have

$$
 E_3(r = 1) = \mu \frac{(E[\beta X e^{-\beta X}] - 1)^2 - E[e^{-\beta X}] E[\frac{\beta^2 X^2}{2} e^{-\beta X}]}{(E[e^{-\beta X}])^3}. \quad (3.16)
$$

Lastly, for $(r = 2, s = 2)$, $Q$ simplifies to

$$
 Q = \begin{pmatrix}
 1 - E[e^{-\beta X}] & 0 \\
 E[(1 - e^{-\beta X})^2] & 1 - E[e^{-2\beta X}]
 \end{pmatrix}.
$$

Hence, we have

$$
 E_2(r = 2) = \mu \frac{E[1 - e^{-\beta X} + e^{-2\beta X}]}{E[e^{-\beta X}] E[e^{-2\beta X}]} \quad (3.17)
$$

Example 3.1. We reconsider Example 2.1, except that now we allow the lifetime $X$ to have an arbitrary absolutely continuous CDF $F$ with density function $f$. We ask the same question: Given that currently we have one repair facility and two spare units, should we (a) buy one more spare unit, or (b) set up another repair facility?
For various choices of the lifetime distributions, we compute MSUT under the current set up \((r = 1, s = 2)\), the modified set up \((r = 1, s = 3)\) under option (a), and the modified set up \((r = 2, s = 2)\) under option (b), using expressions (3.15)–(3.17) respectively. Then using (1.1) and (2.2), we compute the limiting average availability and report them in Table 1 below. In every case, option (b) is preferable to option (a). Note that for every fixed pair \((r, s)\), the limiting average availability is inversely related to the standard deviation of the lifetime distribution, at least within the same family of distributions. Also note that when the lifetime distribution is exponential\((1/50)\) and the repair time distribution is exponential\((1/35)\), we get exactly the same results as in Example 2.1, confirming that Theorem 3.1 is consistent with Theorem 2.1.

Table 1: Limiting average availability \(A_{av}\) of a one-unit system supported by \(r\) repair facilities and \(s\) spare units for various lifetime distributions each with mean 50, when repair time is exponential with mean 35.

| Lifetime Distribution | Standard Deviation | \((r, s)\) |
|-----------------------|-------------------|
|                       |                   | \((1, 2)\) | \((1, 3)\) | \((2, 2)\) |
| exponential\((0.02)\) | 50.0              | .8646     | .9134     | .9578     |
| gamma\((3, 0.06)\)   | 28.9              | .9118     | .9532     | .9826     |
| gamma\((1/3, 1/150)\)| 86.6              | .7855     | .8341     | .9046     |
| lognormal\((3.812, 0.2)\)| 23.5          | .9230     | .9615     | .9877     |
| lognormal\((3.565, \ln 2)\)| 50.0        | .8802     | .9260     | .9697     |
| lognormal\((3.212, 1.4)\)| 87.4          | .8308     | .8776     | .9431     |
| Weibull\((2, \pi /10^4)\)| 26.1      | .9152     | .9560     | .9834     |
| Weibull\((1/2, 0.2)\)| 111.8           | .7697     | .8141     | .8966     |

4. Summary and Conclusions

We have obtained the limiting average availability of a system supported by \(r\) repair facilities and \(s\) spare units when the lifetime is allowed to have an arbitrary continuous
CDF with density function \( f \). We assume the repair times to be exponentially distributed, in order to exploit its lack of memory property in constructing an embedded discrete-time Markov chain. When repair time distribution is other than exponential, except for the case of \((r = 1, s = 1)\), one must keep track of the time on repair of all failed units at all times. Therefore, there is no hope of identifying an embedded discrete-time Markov chain, and the derivation of the limiting average availability will require a technique different from the one presented in this paper. We leave that problem open.

For the case of \((r = 1, s = 1)\), the limiting average availability of the system is given by Sen and Bhattacharjee (1986) as

\[
A_{av} = \frac{E[X]}{E[X \lor Y]} = \frac{\int_0^\infty [1 - H(x, \infty)] \, dx}{\int_0^\infty [1 - H(x, x)] \, dx},
\]

(4.1)

where the (lifetime, repair time) pairs, \((X, Y)\), are IID across the units having a completely arbitrary bivariate CDF \( H \) (with possible dependence within the pair). We present below a streamlined version of their lengthy proof.

For \((r = 1, s = 1)\), we have \( MSUT = E[X] + P\{X \geq Y\} \) MSUT; or equivalently, \( MSUT = E[X]/P\{X < Y\} \) and \( MSDT = E[Y - X]|X < Y\). Hence, by (1.1),

\[
A_{av} = \frac{E[X]}{E[X + P\{X < Y\} E[Y - X]|X < Y]} = \frac{E[X]}{E[X + E[(Y - X) \cdot I\{X < Y\}]]} = \frac{E[X]}{E[X \cdot I\{X \geq Y\}] + E[Y \cdot I\{X < Y\}]} = \frac{E[X]}{E[X \lor Y]}.
\]

(4.2)

The rest of the proof follows from Fubini’s theorem.

In particular, when \( X \) and \( Y \) are independent and \( Y \) has an exponential(\( \beta \)) distribution, we get \( MSDT=E[Y - X]|X < Y] = E[Y] = 1/\beta \) and \( MSUT=E[X]/P\{X < Y\} = \mu /E[e^{-\beta X}] \), which agrees with the result when (3.14) is specialized to \((r = 1, s = 1)\).

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