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Drag of a flexible fiber in a 2D moving viscous fluid *

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Abstract This paper reports the numerical study of the drag of a flexible elastic fiber immersed in a two-dimensional viscous flow using the immersed boundary (IB) method. We found drag reduction of a flexible fiber compared to a stiff one and the drag coefficient decreases with respect to the dimensionless fiber length within a certain range. The results are a starting point for the understanding of the role of flexibility in biological organisms in fluid flows.

Keywords drag reduction, flexibility, soap film, immersed boundary method, fluid-structure interaction, computational fluid dynamics.

Classification numbers: M47.11.+j. 47.15.-x. 47.90.+a.

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I. Introduction

Like rigid objects, flexible objects moving in viscous flows will experience drag. Studying the drag of moving flexible objects in viscous fluids is of significant importance because it may have potential applications in some bioengineering problems. In addition, the hydrodynamic drag many organisms have to bear in rapid flows of water or air may have important influences over their survival. It may affect the body shape and size, the choice of habitat, and more. See Vogel [1] for an overview.

Many organisms have flexible body structures. This flexibility may provide them advantages compared to rigid bodies in fast moving flows. For example, drag may be reduced due to the reconfiguration of the body caused by the forces exerted by flowing fluids. Many researchers have done work along this direction. Denny [2] measured the drag of five differently shaped objects immersed in a flow tank and obtained the probability distribution of drag for each object. The extreme drag was used by the author to study the likelihood of breakage and dislodgment of an organism. Vogel [3] studied experimentally the drag and reconfiguration of a variety of leaves and clusters in turbulent winds. Vogel [4] analyzed the drag, flexibility, and adaptive reconfiguration of sessile organisms. Koehl [5] explored how benthic organisms withstand moving water by use of biomechanics. For more work involving drag and flexibility, see references in [2, 3, 4, 5] and the book by Vogel [1].

Recently an interesting laboratory experiment has been reported by Alben et al. [6]. In that paper the authors performed an experiment—flexible fibres in a flowing soap film and studied the drag reduction of a flexible fibre in viscous flows (with respect to a rigid one) through self-similar bending, both theoretically and numerically. For details of the work and its significance, see [6, 7, 8].

The Reynolds number in [6] is in the range of 2,000 – 40,000. In many biofluid mechanics problems the Reynolds number is on the order of 100\(^1\). Therefore in our work we focus on flows with Reynolds number at range of 12.5 – 375 based on the fibre length, inflow speed, and the soap film viscosity. Our paper reports numerical studies of drag on a flexible fiber immersed in a two-dimensional incompressible flow by the immersed boundary (IB) method. Our simulation is based on a model problem abstracted from the laboratory experiment [6].

Absorbing the most essential material, we build up a model problem from the laboratory experiment [6]. The experiment made use of a 2D soap film tunnel generated by flowing soapy water under gravity from a top reservoir along two supported nylon wires and an immersed flexible glass fibre with its center point tethered otherwise unconstrained. In our model problem the soap film is modeled by an incompressible viscous fluid. The glass fibre is

\(^1\)The spectrum of Reynolds numbers for biological flows is very wide. It ranges from \(10^8\) for a large whale swimming to \(10^{-5}\) for a bacterium swimming. See [1]
treated as a 1D fiber with mass uniformly distributed and the fiber can be bent and stretched/compressed (in simulation the stretch coefficient is chosen such that the fiber experiences almost no stretch). The periodic boundary condition is used on top and bottom (the terminal velocity profile is enforced on top and bottom). The no-slip no-penetration condition is used on left and right. See Fig. 1 for an illustration of our model.

First we assume that the fiber is neutrally buoyant in the soap film and use the FFT based version of IB method [9, 10, 11, 12, 13, 14, 15, 16]. We first computed the drag of the fiber as a function of the inflow speed and its length by varying the inflow speed/length while keeping all the other parameters unchanged. Then we calculated the drag of the fiber as a function of its bending rigidity. Finally we relaxed the neutrally buoyant assumption of the fiber and use the multi-grid [17, 18, 19] based version of IB method [9, 12, 20, 21] to study the influence of the fiber mass density on the drag. In all the cases the drag was calculated by averaging the instantaneous drag computed at \( N \) instants in a time interval \([T_{qs}, T_e]\) (see Section III for detail). The drag coefficient based on the averaged drag was plotted as a function of varied dimensionless parameters. The location and shape of the fiber after time \( T_{qs} \) were drawn together for each varied parameter value.

The remainder of the article is structured as follows: in Section II we address the extraction of an idealized model problem from the laboratory experiment and the IB mathematical formulation of the model problem. Section III talks briefly about the two versions of the IB method that were used in our work for solving the nonlinear system of the integral-differential equations. Section IV first addresses some issues concerning the simulations, and then displays and explains the main simulation results. Finally a concise summary is made to end the article.

II. Mathematical Formulation

Because of the complexity of the actual physical problem\(^2\) involved in the laboratory experiment [6], a model problem is abstracted from the real problem in experiment based on which our numerical study is carried out. The main assumption we make for the model problem is: a) the soap film flow is a 2D Newtonian viscous incompressible flow and can be described by the classic incompressible Navier-Stokes equations (see [22, 23, 24] for details on soap film flows and Navier-Stokes equations). b) the fiber is a 1D material line with zero thickness and totally immersed in the soap film. With these

\(^2\)The soap film thickness is approximately 1 – 3 microns, the diameter of the fibre is approximately 34 microns. Therefore, at microscale the actual physical problem involves a 3D fibre immersed in an almost 2D moving soap film. Given the fact that the fibre is 1 – 5 cm long and the flow tunnel is 9 cm wide and at least 2 meter long, it appears to be a good approximation to treat the problem as two-dimensional. We expect the numerical results reported in this paper should at least qualitatively agree with those of the actual 3D situation.
assumptions our model problem is shown in Fig. 1. (Notice that our proposed model problem is a truly two-dimensional problem, not a problem in which there is no dependence of the flow or the filament displacement on the third direction. See Table 1 for the values and dimensions in 2D of all parameters used in the simulations.) The main soap film flowing tunnel is modeled by a two-dimensional incompressible viscous channel flow (the rectangle domain). The inflow and outflow profile (arrows on top and bottom) are actually taken as in Fig. 2. The flow velocity at left and right sides of the rectangle is zero. A one-dimensional curve (fiber) is used to simulate the flexible elastic fibre. The curve can be bent, compressed, stretched, and may carry mass which is uniformly distributed over the whole length of the curve. The fiber center is attached to a fixed point which is at the middle line of the two side lines and is 4 cm away from the top boundary. The fiber is unconfined everywhere else.

Our IB mathematical formulation of the model problem is as follows:

\[
\rho(x, t) \left( \frac{\partial u}{\partial t} + \frac{1}{2} (u \cdot \nabla u + \nabla \cdot (uu)) \right) = -\nabla p + \mu \Delta u + f(x, t) - \lambda u - \rho(x, t)g \tag{1}
\]
\[ \nabla \cdot \mathbf{u} = 0 \] (2)

\[ f(x, t) = \int \mathbf{F}(s, t) \delta(x - X(s, t)) ds \] (3)

\[ \mathbf{F}(s, t) = -\frac{\partial \varepsilon}{\partial \mathbf{X}} = -\frac{\partial (\varepsilon_s + \varepsilon_b)}{\partial \mathbf{X}} \] (4)

\[ \varepsilon_s = \frac{1}{2} K_s \int (|\frac{\partial \mathbf{X}}{\partial s}| - 1)^2 ds \] (5)

\[ \varepsilon_b = \frac{1}{2} K_b \int |\frac{\partial^2 \mathbf{X}(s, t)}{\partial s^2}|^2 ds \] (6)

\[ \frac{\partial \mathbf{X}}{\partial t}(s, t) = \mathbf{U}(s, t) \] (7)

\[ \mathbf{U}(s, t) = \int \mathbf{u}(x, t) \delta(x - X(s, t)) dx \] (8)

\[ \rho(x, t) = \rho_0 + \int M \delta(x - X(s, t)) ds \] (9)

The classic viscous incompressible Navier-Stokes equations (Eqn. 1 and Eqn. 2) are used to describe the motion of the fluid and the fiber. Notice that the convection term is rewritten in a special form in which the discretization is done directly. The \( \lambda \) is the constant coefficient of air resistance, \( \mathbf{g} \) is the gravitational acceleration vector. The \( f(x, t) \) is the Eulerian force density defined on the fixed Eulerian grid which is calculated from the Lagrangian force density \( \mathbf{F}(s, t) \) defined on the Lagrangian grid by Eqn. 3. Notice that Eulerian force density \( f(x, t) \) is singular along the fiber like a one-dimensional \( \delta \) function. However, the integration of \( f(x, t) \) over any finite region produces a finite value which is equal to the total force over that part of the fiber happening to lie within the integration region.

In Eqn. 3 the function \( \delta(x) \) is the two-dimensional Dirac delta function. The Lagrangian force density \( \mathbf{F}(s, t) \) is defined by Eqn. 4 where the energy \( (\varepsilon) \) generated by stretching/compression \( (\varepsilon_s) \) and bending \( (\varepsilon_b) \) are defined by Eqn. 5 and Eqn. 6, respectively. The total elastic potential energy \( \varepsilon = \varepsilon_s + \varepsilon_b \). The constant \( K_s \) is the fiber stretching coefficient which is chosen in computation so that the fiber has almost no stretch and constant \( K_b \) is the bending rigidity which was measured in the laboratory experiment.

Eqn. 4 is essentially the principle of virtual work. The statement that \( \mathbf{F} = -\frac{\partial \varepsilon}{\partial \mathbf{X}} \) is shorthand for

\[ d\varepsilon = \int (-\mathbf{F}) \cdot d\mathbf{X} ds \] (10)
where $d\mathbf{X}$ is an infinitesimal perturbation of the elastic fiber configuration and $dc$ is the resulting infinitesimal perturbation in the elastic energy. Since $\mathbf{F}ds$ is the force of the element $ds$ of the fiber on the fluid, $(-\mathbf{F})ds$ is the force of the fluid on that element of the fiber. Thus the above expression for $d\varepsilon$ is the work of the fluid on the fiber during the perturbation. The reason this work goes entirely into elastic energy is we have assigned all of the mass of the fiber to the fluid (see Eqn. 9). The fiber itself does not carry any kinetic energy.

The motion of the fiber is described by a system of first order ordinary differential equations, Eqn. 7. The variable $s$ is the Lagrangian coordinate of the fiber, $t$ is the time variable. The $\mathbf{X}$ is the Eulerian coordinate of the fiber whose Lagrangian coordinate is $s$. The fiber velocity $\mathbf{U}(s, t)$ is interpolated from the fluid velocity field $\mathbf{u}(\mathbf{x}, t)$ by the same $\delta$ function as is used to convert the forces from the fiber to the nearby fluid. (See Eqn. 8.)

The Eulerian mass density $\rho(\mathbf{x}, t)$ is defined by Eqn. 9 which accounts for both the mass of the fluid and that of the immersed fiber if the fiber is not neutrally buoyant in the fluid. Here $M$ is the Lagrangian mass density of the fiber (mass per unit length). Like the Eulerian force density $f(\mathbf{x}, t)$, the Eulerian mass density $\rho(\mathbf{x}, t)$ is also singular along the fiber akin to a one-dimensional $\delta$ function supported only along the fiber. Also integration of $\rho(\mathbf{x}, t)$ performed over any finite region gives a finite result: the total mass that lies within that region. Note that this includes a contribution from the fiber and a contribution from the fluid.

Although the mass density $\rho(\mathbf{x}, t)$ is variable the fluid is still incompressible. That is to say

$$\frac{D\rho(\mathbf{x}, t)}{Dt} = \frac{\partial \rho(\mathbf{x}, t)}{\partial t} + \mathbf{u}(\mathbf{x}, t) \cdot \nabla \rho(\mathbf{x}, t) = 0$$

still holds after the fiber mass is spread into the fluid. For proof see [9] [20].

The steady state velocity field of the flow without the fiber is $\mathbf{u}(\mathbf{x}, 0) = (0, v_0(x))$ where $v_0(x)$ is the flow terminal velocity profile which solves the following boundary value problem:

$$\begin{cases}
\mu v_{xx} - \lambda v - \rho_0 g = 0 \\
v(x_1) = v(x_2) = 0
\end{cases}$$

where $x_1$ and $x_2$ are the $x$-coordinates of the position of the two left and right lines. The above DE is obtained by setting $\mathbf{u} = (0, v(x))$, $\frac{\partial}{\partial t} = 0$, $p(\mathbf{x}, 0) = \text{constant}$ in the incompressible Navier-Stokes equations. The solution to the boundary value problem is:

$$v_0(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} - V_0$$
where $V_0$ is the film terminal speed\(^3\) which was measured in the laboratory experiment. Note that we have eliminated the unknown air resistance coefficient $\lambda$ in favor of the measurable parameter $V_0$.

\[
r_1 = \sqrt{\frac{\rho g}{V_0 \mu}}, \quad r_2 = -\sqrt{\frac{\rho g}{V_0 \mu}}
\]

\[
C_1 = \frac{V_0 (e^{r_2 x_2} - e^{r_2 x_1})}{e^{r_1 x_1 + r_2 x_2} - e^{r_1 x_2 + r_2 x_1}}, \quad C_2 = \frac{V_0 (e^{r_1 x_1} - e^{r_1 x_2})}{e^{r_1 x_1 + r_2 x_2} - e^{r_1 x_2 + r_2 x_1}}
\]

See Fig. 2 for a typical solution.

![Typical initial velocity profile](image)

Figure 2: The initial velocity and inflow velocity profiles for inflow speed 200 cm/s. The x-axis represents the flow tunnel width direction. The y-axis is the 2\(^{nd}\) component (i.e. $v(x)$) of the velocity $u(x, 0)$. (Note $v$ is a function of $x$ only here and the 1\(^{st}\) velocity component is zero.) Note that the inflow velocity profile is a flat line on most part of the channel except near the left and right boundaries; this is because of the air resistance. This profile is imposed as the initial condition and inlet/outlet boundary conditions for velocity.

The value of $X(s, 0)$ is specified as the initial condition for the fiber and the boundary condition is that $X(s_c, t)$ is constant. Here $s_c$ is the Lagrangian coordinate of the fiber center point. The fluid velocity profile $(0, v_0(x))$ is

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\(^3\)The film terminal speed is the maximum value of the film terminal velocity which is defined as the vertical component of the velocity of the flowing soap film without introduction of the elastic fibre, i.e., the solution to the above ODE.
specified at inflow and outflow and the fluid velocity is equal to zero on the two side wires. The initial condition for the flow is that the velocity field is given as \((0, v_0(x))\). See Fig. 2.

III. Numerical Methods

Our mathematical formulation (the IB formulation) in Section II is a nonlinear system of integral-differential equations which can be numerically solved by two different versions of the IB method. First we suppose the fiber is neutrally buoyant in the fluid. Then the Eulerian mass density \(\rho\) is a constant in space and time. The discrete Fast Fourier Transform can be used to solve the discretized incompressible Navier-Stokes equations (see \([10, 11, 12]\) for details of this approach). If the fiber is not neutrally buoyant in the surrounding fluid, then the above method does not work any longer. Instead the multi-grid version of the IB method \([9, 12, 21]\) is used. The Navier-Stokes equations are discretized by a projection method and the resulting linear system is solved by a multi-grid V-cycle scheme. For detail see \([20, 21]\). Notice that in both cases the convection term is linearized and a skew-symmetrical treatment is employed. This guarantees the conservation of fluid kinetic energy.

IV. Numerical Results

The parameters used in our simulation are tabulated in Table 1. The range of some non-dimensional parameters are tabulated in Table 2. Refer to Table 1 for the symbol’s uses in definition of the nondimensional parameters in Table 2.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>soap film terminal speed ((V_0))</td>
<td>(50 - 300 \text{ cm/sec})</td>
</tr>
<tr>
<td>soap film kinematic viscosity ((\nu))</td>
<td>(4 (\text{cm}^2/\text{sec}))</td>
</tr>
<tr>
<td>soap film density ((\rho_0))</td>
<td>(3 \times 10^{-4} \text{ g/cm}^2)</td>
</tr>
<tr>
<td>filament length ((L))</td>
<td>(1 - 5 \text{ cm})</td>
</tr>
<tr>
<td>filament linear density ((M))</td>
<td>(0 - 2.8 \times 10^{-4} \text{ g/cm})</td>
</tr>
<tr>
<td>filament bending rigidity ((K_b))</td>
<td>(0.28 - 84 \text{ erg \cdot cm})</td>
</tr>
<tr>
<td>gravitational acceleration ((g))</td>
<td>(980 \text{ cm/sec}^2)</td>
</tr>
<tr>
<td>width of the film ((W))</td>
<td>(9 \text{ cm})</td>
</tr>
<tr>
<td>height of the film ((H))</td>
<td>(18 \text{ cm})</td>
</tr>
</tbody>
</table>

Table 1: Parameters used the simulations

Before the simulation results are reported, we need to address the following issues: flow visualization, flow unsteadiness, drag average, and refinements of mesh width and time step size.

The flow is visualized by inserting massless markers at the inlet boundary. The markers are advected by flowing soap film and their velocity is interpolated from the nearby fluid velocity by the same smoothed delta function.
<table>
<thead>
<tr>
<th>Name</th>
<th>Definition</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reynolds Number ($Re$)</td>
<td>$\frac{V_0 L}{\nu}$</td>
<td>$12.5 - 375$</td>
</tr>
<tr>
<td>Fibre Length ($L$)</td>
<td>$\frac{L}{\nu}$</td>
<td>$0.1111 - 0.5556$</td>
</tr>
<tr>
<td>Fibre Mass Density ($M$)</td>
<td>$\frac{M}{\rho_0 L}$</td>
<td>$0 - 0.284$</td>
</tr>
<tr>
<td>Froude Number ($Fr$)</td>
<td>$\frac{V_0^2}{gL}$</td>
<td>$0.773 - 27.83$</td>
</tr>
<tr>
<td>Bending Rigidity ($K_b$)</td>
<td>$\frac{K_b}{\rho_0 V_0^2 L^3}$</td>
<td>$0.0012 - 0.415$</td>
</tr>
</tbody>
</table>

Table 2: Non-dimensional parameters used in the simulations. For the meaning of symbols used in definition of the non-dimensional parameters, see Table 1.

that is used to interpolate the velocity of the fibre. Vorticity contours are utilized to illustrate the vorticity field. A typical picture is shown in Fig. 3. The fibre is 3.3 cm long with a bending rigidity $2.8 \times 10^{-9}$ Jm. The inflow speed is 200 cm/s. The fibre is assumed to be neutrally buoyant in the soap film. The top panel plots the instantaneous positions of the markers for 4 different yet close instants and the bottom panel plots the vorticity contours at corresponding times.

It can be seen from an animation of the simulations that as a response to the flowing down soap film, the tethered flexible fibre adjusts its position to be aligned with the main stream flow by bending itself with respect to the fixed center point. The fibre self-bending due to flexibility may have significant impact on the dynamics of the flow-fibre system. The self-bending decreases the effective fibre length and causes the wake region behind to narrow. It may also cause the vorticity shed to interact with each other and make the wake flow more complex. The interaction of the fibre flexibility and the vortex shedding is a complex and interesting issue which deserves further future study. The boundary layer of the fibre is evident in Fig. 3. The thickness of the upper boundary layer (on the upstream side of the fibre) is approximately 3 mm, which is 0.091× fibre length and about 8× mesh width. The boundary layer on the concave side of the fibre is quite chaotic and difficult to quantify.

From Fig. 3 we can see that the flow is actually not steady, the wake oscillates (see top panel). The oscillation of the wake is probably caused by the vortex shedding from the two free fibre ends which incurs the small-amplitude vibration of the fibre. As a consequence, the drag the fibre experiences may vary with time as shown in Fig. 4. (See the caption of Fig. 4 for the parameter values of the three cases.) The top figures plot the drag versus time, the bottom figures plot the fibre position and shape (dotted lines for $t \leq T_{qs}$ and solid lines for $t \geq T_{qs}$ with the time interval between two adjacent plots $T_s$). The worst scenario (the most unsteady case) is shown on the left most two figures (top and bottom). The thickness of the solid dark line indicates
the oscillation range. The fibre vibrates within this range after $T_{qs}$. The maximum oscillation amplitude is approximately 1.5 mm which is $0.0455 \times$ fibre length. The other two panels show two representative scenarios for most of the simulation outcomes reported here. Notice that for these cases the oscillation range is too small to be visible in the figures. The maximum oscillation amplitude is less than 1% of the fibre length.

To obtain a qualitative behavior of the drag with respect to some parameters like inflow speed, we compute an averaged drag over $N$ equally spaced instances between time $T_{qs}$ and $T_t$ to remove the unsteadiness in instantaneous drag. The $T_{qs}$ and $T_e$ are the starting and ending instants of drag average respectively. The time $T_{qs}$ is chosen such that for most of the cases the fibre has reached a small-amplitude periodic oscillation state ("quasi-steady" state for shorthand), which can be seen from the drag versus time plot (either the middle or the right panel of Fig. 4). The simulation terminal time $T_e$ and the time spacing between two neighboring instantaneous drag computations $T_s$ are chosen arbitrarily. In this work $N = 200$, $T_{qs} = 0.1 \, s$, which is approximately $5 \times$ the characteristic time scale $T_e = V_0/L$, $T_e = 0.2 \, s$, and $T_s = 5 \times 10^{-4} \, s$.

The instantaneous drag the fibre experiences may be defined as the integral of the Lagrangian forces in vertical direction ($y$-direction) along the fibre. In our computation the drag is computed as follows: Let $L_f(1 : n)$ ($1 : n$ represents 1, 2, ..., $n$) be the Lagrangian force density in the vertical direction ($y$-direction) at each fibre grid point at each recorded time. The summation of $L_f(1 : n)$, denoted by $D$, gives us the instantaneous drag at each time, and the average of $D$ over $N$ instants gives us the average drag $\bar{D}$ the fibre experiences after time $T_{qs}$. The drag per unit length was drawn as a function of inflow speed and the drag coefficient $C_D = \frac{D}{\rho_0 V_0^2 L}$ was drawn versus the dimensionless fibre length, bending rigidity, and mass density respectively.

All the simulation results presented below are obtained on a grid with resolution $256 \times 512$ (the finest grid if the multigrid is used) and a time step size $\Delta t = 10^{-6} \, s$. For some cases the results such obtained were compared to those on a grid $64 \times 128$, $128 \times 256$, and $512 \times 1024$ with $\Delta t = 10^{-6} \, s$ and those on a grid $256 \times 512$ with $\Delta t = 4 \times 10^{-6} \, s$, $\Delta t = 2 \times 10^{-6} \, s$, and $\Delta t = 0.5 \times 10^{-6} \, s$. See Fig. 5 for a typical picture. (The fibre is neutrally buoyant and 2 cm long with a bending rigidity $2.8 \times 10^{-9} \, J/m$. The inflow speed is 150 cm/s. Both FFT and multigrid versions are used. Results shown in Figure 5 are from FFT. Results from the multigrid version match those from FFT extremely well if strict enough convergence criterion is used for multigrid iteration.) The upper figure plots the drag versus time for results on the four grids (see the Figure caption for detail) with the same time step size $10^{-6} \, s$. The lower figure plots the drag versus time for results on a grid ($256 \times 512$) with four different time step size (see the Figure caption for detail). The four drag-time curves in the lower figure are almost not
discernable. The curves appear to be convergent in the upper figure. To see the difference between results obtained from the grid $256 \times 512$ with $\Delta t = 10^{-6} \, s$ (all results reported in this paper are obtained from such a grid spacing and $\Delta t$) and next finer grid and next smaller time step size (by a factor of 2), we notice that the two curves in the lower figure (dashed and solid lines) are almost not distinguishable. The maximum relative difference is 0.25\% during $T_{qs} \leq t \leq T_e$ and the relative difference in the averaged drag is 0.10\%. In the upper figure, The results obtained from two grids ($256 \times 512$ and $512 \times 1024$) with the same time step size $\Delta t = 10^{-6} \, s$ are discernable with a maximum relative difference of 5.3\% during $T_{qs} \leq t \leq T_e$, and a relative difference of 3.1\% in the average drag. To quantify the convergence rate, we computed the ratio $\frac{\|V_{2h} - V_{4h}\|_{L_2}}{\|V_h - V_{2h}\|_{L_2}}$ (where $V$ is the vertical velocity component, $h$ is mesh size, $L_2$ is the $L_2$ norm) using the results on above four successively refined grids. The value was 2.37 and 2.46 for the FFT version and 2.35 and 2.42 for the multigrid version. (This seems to indicate a convergence rate of approximately 1.3 for both methods.) Because of computational limitations we are not able to perform such comparisons for each case reported here. We believe the simulations on a grid $256 \times 512$ with $\Delta t = 10^{-6} \, s$ can give us reasonably good results which are at least qualitatively correct.

Our numerical study is presented in the following order: first the simulation result of drag with respect to varied inflow speed for two fibers with very different bending rigidity is presented, then the results of drag coefficient as a function of dimensionless fiber length is addressed. Thirdly the results on drag and fiber bending rigidity is given. Finally the assumption that the fiber is neutrally buoyant in ambient fluid is relaxed and the fiber mass density is artificially set to different values and the influence of fibre mass on drag is reported. In all the cases, we only vary one parameter at a time while all the others are fixed. In each case the shapes of the fiber were recorded and plotted together for each value of the varied parameter.

The left figure of Fig. 6 compares the drag per unit length as a function of inflow speed for two fibres with exactly the same parameters except the bending rigidity. The circle (o) corresponds to a fibre with stiffness $8.4 \times 10^{-8}$ J m, the plus sign (+) corresponds to a fibre with stiffness $2.8 \times 10^{-9}$ J m which is 30 times smaller than the stiff fibre. We can see from the plot that the flexible fibre experiences drag reduction compared to the stiff one. The other two figures in Fig. 6 show the position and shape of each fibre after it reaches the quasi-steady state in a flowing soap film. Each curve represents the position and shape of the fibre after $T_{qs}$ in a soap film with a different inflow speed ranging from $0.5$ – $3.0$ m/s. It is seen from these plots that a flexible fibre can more easily get bent and streamlined with the main flow direction, thus to avoid drag. This may explain why a flexible fibre has much less drag than a stiff one in the same flowing soap film. It seems that the
drag reduction is caused mainly by the shape of a fibre.\(^4\) The drag reduction induced by the bending and streamlining of a flexible object can be beneficial for many organisms living in swept waters or winds. For example, it may help the trees of broad leaves to survive in strong winds (see Vogel [3, 7]), or the giant bull kelp (a large seaweed) to avoid breakage in fast currents (see Koehl [5]).

Even though the Reynolds numbers in our work are approximately 200 times lower than those in the work by Alben et al. [6], it is interesting to point out that the drag in Alben’s case and ours is roughly at the same order of magnitude ($2 - 11$ dyn/cm for the flexible case and $2 - 52$ dyn/cm for the stiff case). The drag value for a flexible fibre is quite close in Alben’s work and ours. For example, the drag per unit length of the flexible fibre at inflow speed $300$ cm/s is both approximately $11$ dyn/cm. The drag difference is quite wide in the case of a stiff fibre. For instance, at inflow speed $266$ cm/s the drag is $29$ dyn/cm in Alben’s case and $43$ dyn/cm in ours. It is also noticed that the drag versus inflow speed is quite different in Ablen’s work (a nonlinear curve) and ours (nearly a straight line) for the stiff fibre. The two curves are closer for the flexible fibre. The position and shape of the fibre here look qualitatively similar to those in Alben’s paper [6]. We believe that the significant difference in Reynolds numbers and the nonlinearity in the interaction of the soap film and the flexible fibre may account for the quantitative disparity in drag and fibre shape in the two works.

Fig. 7 demonstrates the drag coefficient and total drag as a function of fibre length. The top figure in Fig. 7 exhibits the drag coefficient with respect to its dimensionless length $\tilde{L}$. The middle figure displays the total drag with respect to its length $L$. The bottom figure demonstrates how the fibre position and shape varies with its length. It is interesting to notice that the total drag a fibre bears increases first as a function of fibre length but decreases later as the length continues to increase. The drag coefficient is a decreasing function of dimensionless fibre length $\tilde{L}$. The drag reduction induced by length may be explained by the bottom figure that shows the location and shape of the fibre after $T_{qs}$ for different values of fibre length ranging from $1 - 5$ cm. We can see from this figure that a long fibre can be bent and streamlined in the flow more easily than a short one and thus experiences less drag. Many organisms living in waters may take advantage of this by developing a thin and long body to minimize drag [1].

Fig. 8 plots the drag coefficient as a function of non-dimensional fibre bending rigidity $\tilde{K}_b$. The left figure is the drag coefficient $C_D$ as a function of bending rigidity $\tilde{K}_b$ and the right figure is the position and shape of the fibre after $T_{qs}$. As the bending rigidity increases the drag increases swiftly at

\(^4\)The other possible factor for drag reduction is the transition from laminar flow to turbulence. This seems to be unlikely for our case, because the maximum Reynolds number is 375.
the beginning, but the increase slows down later. This may be explained by
the right figure that shows the position and shape of the fibre in quasi-steady
state. From this figure we see that the fibre location and shape changes
quickly as the bending rigidity increases at the beginning, but does not differ
much after the bending rigidity is large enough to keep the fibre nearly a
horizontal line in the flowing soap film.

Fig. 9 shows the influence of fibre mass on the drag. The left figure is
the drag coefficient as a function of non-dimensional fibre linear mass density
\( M \). In simulation, the parameter of fibre linear mass density is artificially
and gradually increased from zero to \( 2.8 \times 10^{-5} \) kg/m. We find that the fibre
mass does not have an important influence on the drag coefficient. It may
be expected that the fibre mass can result in drag reduction by helping the
fibre bend as a result of gravity and get streamlined with the mainstream
direction. According to the simulation results this effect is not significant at
least at the range \([0, 2.8 \times 10^{-5}]\) kg/m.

V. Summary

Motivated by the biological consequences of the drag organisms encounter
in swiftly moving fluids, we numerically studied the drag a flexible fiber expe-
riences in a 2D viscous incompressible flow. If the fiber is neutrally buoyant
in the flow, the FFT based IB method is used. If the fiber is not neutrally
buoyant, the multi-grid version of IB method is adopted. The drag as func-
tions of inflow speed, fiber length, bending rigidity, and fiber mass density
were computed and plotted respectively and the locations and shapes of the
fiber were recorded and drawn in each case. For a fibre in a flowing soap
film we numerically found that 1) A flexible fibre experiences drag reduction
compared to a stiff one. 2) Fibre mass has little effect on the drag at a certain
range. 3) Bending rigidity has a significant influence on drag. 4) Drag coeffi-
cient is a decreasing function of the dimensionless fibre length at a certain
range.

Because of the high complexity of the actual physical situation in the lab-
atory experiment, our simulation is performed based on an idealized 2D
model problem which does not take into account the finite thickness of the
soap film and the fibre diameter (both small compared to other dimensions).
We believe these numerical results should remain valid in the actual 3D situa-
tion at least qualitatively. Quantitative difference is possible. Our simulation
results may be used to explain some biological phenomena closely connected
to hydrodynamic drag organisms have to bear in rapidly flowing fluids.

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References


Figure 3: The flow visualization for a typical simulation. The top shows the positions of the fluid markers at 4 different instants. The bottom shows the contours of vorticity at corresponding times. The 4 time instants are (from the left to right) 0.1640, 0.1760, 0.18880, 0.2, respectively. (unit: second) The fibre is 3.3 cm long with a bending rigidity $2.8 \times 10^{-9}$ Jm. The inflow speed is 200 cm/s. The fibre is neutrally buoyant.
Figure 4: The time variation of drag and fibre position for three cases. The top group plots the drag as a function of time. The bottom group plots corresponding fibre positions. The fibre position is plotted by dotted lines before time $T_{qs}$ and by solid lines after $T_{qs}$. Notice that the thickness of the solid line indicates the oscillation range of the fibre. The left panel (top figure for drag and bottom for fibre position) is the most unsteady case among all the simulations reported in this paper. (The fibre is 3.3 cm long with the bending rigidity $2.8 \times 10^{-9} \text{ Jm}$. The inflow speed is 300 cm/s. The fibre is neutrally buoyant.) Both the drag and the fibre position vary with time obviously. The maximum oscillation amplitude is 1.5 mm. The middle and right panels show two typical scenarios which represent most of the simulation results. The fibre in the middle figure is 3.3 cm long with the bending rigidity $2.8 \times 10^{-9} \text{ Jm}$. The inflow speed is 150 cm/s. The fibre mass density is $5 \times 10^{-6} \text{ kg/m}$. The fibre in the right figure is 5 cm long with the bending rigidity $2.8 \times 10^{-9} \text{ Jm}$. The inflow speed is 150 cm/s. The fibre is neutrally buoyant.
Figure 5: Typical results of mesh width and time step size refinement. The upper figure plots the drag versus time for results on four grids (big dots, soon looking like a thick solid line on the top, for $64 \times 128$; small dots for $128 \times 256$; dashed line for $256 \times 512$; and solid line for $512 \times 1024$) with the same time step size $10^{-6}$ s. The drag-time curves appear to be convergent when the space mesh is refined when $\Delta t$ is kept very small to minimize the error from discretization in time. The lower figure plots the drag versus time for results on a grid ($256 \times 512$) with four gradually refined time step size. $\Delta t = 4 \times 10^{-6}$ s (dotted line), $\Delta t = 2 \times 10^{-6}$ s (dashdot line), $\Delta t = 10^{-6}$ s (dashed line), $\Delta t = 0.5 \times 10^{-6}$ s (solid line). The four drag-time curves are almost not discernable when $\Delta t$ is thus refined on a fixed fine space grid ($256 \times 512$).
Figure 6: The left figure shows the drag per unit length as function of inflow speed for two fibers with different bending rigidity ("o" is $8.4 \times 10^{-8}$ J m, "+" is $2.8 \times 10^{-9}$ J m). We can see that compared to the stiff fiber the flexible one has less drag. The middle and right figures show the locations and shapes of the fibers in the quasi-steady state. Each curve corresponds to a specific value of inflow speed. The inflow speed varies from 0.5 m/s to 3 m/s. Many curves overlap in the middle figure.
Figure 7: The top figure plots the drag coefficient as a function of dimensionless fiber length. The middle figure plots the total drag as a function of fiber length. The bottom figure plots the position and shape of the fiber for different length. Each curve represents a specific value of length. The length varies from 1 to 5 cm.
Figure 8: The drag coefficient as a function of dimensionless fiber bending rigidity (left) and position and shape of the fiber at quasi-steady state for different values of bending rigidity (right). The bending rigidity ranges from $2.8 \times 10^{-10}$ to $8.4 \times 10^{-8}$ J m. Many curves overlap in the right figure.
Figure 9: The drag coefficient as a function of dimensionless fiber mass density (left) and the position and shape of the fiber in quasi-steady state (right). The mass density ranges from zero to $2.8 \times 10^{-5}$ kg/m. Many curves overlap in the right figure.