Pseudohyperbolicity and the Rapid Decay Property
by
Ronghui Ji and Bobby Ramsey

PR # 06-02

This manuscript and other can be obtained via the
World Wide Web from www.math.iupui.edu

January 2006
Pseudohyperbolicity and the Rapid Decay Property

Ronghui Ji and Bobby Ramsey
Department of Math. IUPUI

1 Introduction

The rapid decay property for a group (known as property RD) was introduced by Jolissaint [J] which generalizes Haagerup’s inequality for free groups [Haa]. This property for groups has deep implications in the analytical, topological and geometrical aspects of the groups. In his thesis [J], Jolissaint proves that groups of polynomial growth and classical hyperbolic groups are of property RD, and the only amenable discrete groups that are of property RD are groups of polynomial growth. de la Harpe [H] improves Jolissaint results and showed that the word hyperbolic groups of Gromov have property RD as well and this leads to the work of Connes and Moscovici [CM] that word hyperbolic groups satisfy the Novikov conjecture. Since then many important works have been done on establishing property RD, notably the works of Lafforgue [La], Chatterji and Ruane [CR], and Drutu and Sapir [DS]. The main obstacle in establishing that a group is of property RD is the verification the Haagerup inequality, which estimates the $C^*$-algebra norm of elements in the complex group algebra.

In this note we will try see how far the hyperbolic argument of de la Harpe can be pushed to include a class of groups wider than just the hyperbolic ones, that have property RD. This will be accomplished by weakening Gromov’s definition of hyperbolicity [G] by replacing the word length metric by an arbitrary left-invariant metric on the group satisfying a weak geodesic property, and weakening the hyperbolic constant to one that depends on the quadrilateral or triangle under consideration. It turns out that under these weakened conditions we can still establish the property RD. We will call
this class of groups \textit{pseudohyperbolic groups} (see the definition in section 4). Our argument is based on another proof of property RD for word hyperbolic groups by Ozawa and Rieffel [OR]. It is desirable to find a concrete group that is pseudohyperbolic but not hyperbolic. So far we have not been able to do so.

Given a group with a length function, it is possible to define the set of rapidly decaying functions on the group, with respect to that length function. This, in turn, leads to the concept of the RD property of a group with respect to the length function. In the past, this property has been approached almost exclusively through the word function on finitely generated groups.

It has been known for some time that hyperbolic groups and groups of polynomial growth both have this RD property. The proofs of these two cases can be unified into a single, very geometric argument. In section 2 we define length functions on groups. In section 3 we introduce the RD property, and give some examples. In section 4 pseudohyperbolic groups are introduced, and in section 5 we extend this to relatively pseudohyperbolic groups.

2 Length Functions

\textbf{Definition 2.1.} Let \( G \) be a countable discrete group. A length function on \( G \) is a map \( \ell : G \to \mathbb{R}_+ \), the nonnegative real numbers, satisfying the following conditions:

1. \( \ell(g) = 0 \), if and only if \( g = e \), the identity element of \( G \).
2. \( \ell(gh) \leq \ell(g) + \ell(h) \) for all \( g, h \in G \).
3. \( \ell(g^{-1}) = \ell(g) \) for all \( g \in G \).

For a length function \( \ell \), we denote \( E_m^\ell = \{ g \in G : \ell(g) = m \} \), and \( B_m^\ell = \{ g \in G : \ell(g) \leq m \} \). When there is no chance of confusion, we will denote these by \( E_m \) and \( B_m \) respectively.

\textbf{Example 2.1.} Let \( G \) be generated by a finite generating set \( S \). For convenience, we will assume that \( S \) is symmetric, that is \( S^{-1} = S \). For any \( g \in G \) we will define \( \|g\|_S = \min \{ k : g = s_1 s_2 \ldots s_k, s_i \in S \} \). This is the algebraic word-length function of \( G \) induced by the generating set \( S \). When the generating set is clear, we will use \( \|g\| \) instead of \( |g|_S \).
Example 2.2. Let \((X, d)\) be a metric space, and let \(G\) be a group acting freely by isometries on \(X\). For each \(x_0 \in X\), define the function \(\ell_{x_0}(g) = d(x_0, gx_0)\) for all \(g \in G\). This is the length function on \(G\) induced by the action. Notice that this length depends on the initial point \(x_0\).

To illustrate the difference between these two types of length functions, consider \(\mathbb{Z}^2\), with symmetric generating set \(S = \{(0, 1), (1, 0), (0, -1), (-1, 0)\}\). For \((n, m) \in \mathbb{Z}^2\), we have in this word-length function \(|(n, m)|_S = |n| + |m|\), where \(|n|\) and \(|m|\) are the absolute values. Consider now \(\mathbb{Z}^2\) acting on \(\mathbb{R}^2\), in the standard euclidean metric, by translations. Let our base point be the origin, so that \((m, n)(0, 0) = (m, n)\). Thus \(\ell((m, n)) = \sqrt{m^2 + n^2}\).

With the above properties, to any length function \(\ell\), one is able to associate a metric \(\rho\) by setting \(\rho(g, h) = \ell(g^{-1}h)\). Notice that this metric is invariant with respect to the left action of \(G\) on itself, that is to say \(\rho(\gamma g, \gamma h) = \rho(g, h)\) for any \(g, h, \gamma \in G\). Thus with a length function, we can view our group \(G\) as a metric space \((G, \rho)\).

Definition 2.2. We say that a length function \(\ell_1\) dominates a length function \(\ell_2\) if there are constants \(a, b > 0\) such that for any \(g \in G\) we have \(\ell_2(g) \leq a\ell_1(g) + b\). Two length functions, which dominate each other, are said to be equivalent.

Let us return to the Example 2.1 of the algebraic word-length. Let \(G\) be a countable, discrete group, with symmetric finite generating sets \(S\), and \(S'\), yielding word-length functions \(\cdot |_S\) and \(\cdot |_{S'}\) respectively. As the generating sets are different, these length functions, and the metric functions they induce, are different. They are, however, equivalent length functions, and the identity map \(Id : (G, \rho_S) \rightarrow (G, \rho_{S'})\) is a quasi-isometry.

3 The RD Property

The left-regular representation of a group \(G\) gives an action of \(G\) on the space of square-summable functions \(l^2G\), by \(yh(x) = h(y^{-1}x)\) for all \(y \in G, h \in l^2G\). This representation extends to \(\mathbb{C}G\) by \(f(\xi)(x) = \sum_{g \in G} f(g)\xi(y^{-1}x)\) for \(f \in \mathbb{C}G, \xi \in l^2G\). Thus, \(\mathbb{C}G\) acts on \(l^2G\) by convolution. This gives an embedding of the group algebra, \(\mathbb{C}G\), into \(B(l^2G)\), the bounded linear operators on the Hilbert space \(l^2G\). The reduced group \(C^*\)-Algebra of \(G\), denoted \(C_r^*G\), is the operator norm-closure of \(\mathbb{C}G \subset B(l^2G)\).
Definition 3.1. A countable discrete group $G$ is said to have the RD property, with respect to the length function $\ell$, if the space of rapidly decreasing functions lies in the reduced group $C^*$-Algebra. That is, if $H^\infty_\ell(G) \subset C^*_r(G)$.

The condition $H^\infty_\ell(G) \subset C^*_r(G)$ is equivalent to the following condition: There exists a polynomial $Q$ such that for any $f, \xi \in \mathcal{C} G$ if $\sup \mathcal{p} f \subset E_k$, and supp $\xi \subset E_n$ then $\|P_m(f \ast \xi)\|_2 \leq Q(k)\|f\|_2\|\xi\|_2$, where $P_m : i^2 \mathcal{G} \to i^2 E_m$ is the standard projection. Denote by $D^\ell_{\ell,\sigma}(x) = \{(\bar{x}, \hat{x}) \in G^2 : x = \bar{x} \hat{x}, \ell(\bar{x}) \leq a + \sigma, \ell(\hat{x}) \leq (\ell(x) - a) + \sigma\}$ We will use the notation $D^\ell_{\ell}(x)$ when there is no chance of confusion.

Definition 3.2. A length function $\ell$ on a group $G$ is $\sigma$-quasiconvex, for a constant $\sigma > 0$, if for every $x \in E_k$ and every $a \leq k$, the sets $D^\ell_{\ell,\sigma}(x)$ are nonempty, and every ball of finite radius has finite cardinality.

Notice that for any finitely generated group, a word length function is always 1-quasigeodesic. More generally if, in the metric induced by the length function $\ell$, the group is a proper, weakly-geodesic metric space, in the sense of [KS], then the length function is quasiconvex. The terminology is suggested by the following example:

Example 3.1. Let $\Gamma$ be a finitely generated group with word-length function $| \cdot |$, and let $G$ be a subgroup of $\Gamma$. If $G$ is quasiconvex as a subspace of the geodesic metric space $\Gamma$, then the word-length function restricted to $G$ is a quasiconvex length function.

Definition 3.3. Let $G$ be a countable group, and $\ell$ a length function on $G$. Denote by $\{a_n\}_{n=0}^{\infty}$ the countable set of values attained by the length function, ordered such that $0 = a_0$ and $a_i < a_{i+1}$. The length function $\ell$ is said to be uniformly discrete if there is a constant $\kappa$ such that $|a_{i+1} - a_i| > \kappa$ for all $i \geq 0$.

Definition 3.4. A metric space $(X, d)$ is said to be hyperbolic if there is a constant $\delta > 0$ such that for any four points $w, x, y, z \in X$ we have that:

$$d(w, x) + d(y, z) \leq \max \{d(w, y) + d(x, z), d(w, z) + d(x, y)\} + \delta$$

This definition of a hyperbolic metric space is equivalent to the definition in terms of the Gromov product, and to the definition in terms of the thin triangles condition [GH].
In [J], Jolissaint showed that all classical hyperbolic groups had property RD. In [H], de la Harpe showed that all Gromov hyperbolic groups had property RD.

**Theorem 3.1.** Let $G$ be a countable group with a uniformly discrete $\sigma$-quasiconvex length function, $\ell$. If $G$ is hyperbolic in the metric induced by $\ell$, then $G$ has the RD property.

**Proof.** Let $\{a_n\}_{n=0}^\infty$ be the set of values attained by the length function, ordered as above, with uniform discreteness constant $\kappa$, hyperbolicity constant $\delta$. Denote by $\rho$ the metric induced by $\ell$.

Let $f, \xi \in \mathbb{C}G$ with $\text{supp} \ f \subset E_{a_k}$ and $\text{supp} \ \xi \subset E_{a_n}$, and $x \in E_{a_m}$. Let $q = (a_n + a_k - a_m)/2$. Then:

$$f * \xi (x) = \sum_{x=yz} f(y)\xi(z) = \sum_{x=yz \in E_{a_k}, z \in E_{a_n}} f(y)\xi(z)$$

As $\ell$ is quasiconvex, $D^\sigma_{a_n-q}(x)$ is nonempty for each $x \in E_{a_m}$. Thus for each $x$ we will fix once and for all, one pair $(\bar{x}, \bar{\bar{x}}) \in D^\sigma_{a_n-q}(x)$. If $x = yz$ as in the sum, examine the quadruple $(e, x, \bar{x}, y) \in G^4$:

$$\rho(e, x) + \rho(\bar{x}, y) \leq \max \{\rho(e, \bar{x}) + \rho(x, y), \rho(e, y) + \rho(x, \bar{x})\} + \delta$$

$$a_m + \rho(\bar{x}, y) \leq \max \{(a_k - q + \sigma) + a_n, a_k + (a_n - q + \sigma)\} + \delta$$

$$\rho(\bar{x}, y) \leq \max \{(a_k + a_n - a_m) - q + \sigma, (a_k + a_n - a_m) - q + \sigma\} + \delta$$

Thus, for $u = \bar{x}^{-1}y$ we have $y = \bar{x}u$ with $\ell(u) \leq q + \sigma + \delta$. This also yields a decomposition $z = u^{-1}\bar{x}$. We then have:

$$f * \xi (x) = \sum_{u \in G, \ell(u) \leq q + \sigma + \delta} f(\bar{x}u)\xi(u^{-1}\bar{x})$$

For a given $y \in E_{a_k}$, how many factorizations of the form $y = su$ with $\ell(s) \leq a_k - q + \sigma$ and $\ell(u) \leq q + \sigma + \delta$ are there? Assume that $y = su = tv$ with $\ell(s), \ell(t) \leq a_k - q + \sigma$ and $\ell(u), \ell(v) \leq q + \sigma + \delta$. Consider the quadruple
\((e,y,s,t) \in G^4:\)

\[
\rho(e,y) + \rho(s,t) \leq \max \{\rho(e,s) + \rho(y,t), \rho(e,t) + \rho(y,s)\} + \delta
\]

\[
a_k + \rho(s,t) \leq \max \{(a_k - q + \sigma) + (q + \sigma + \delta), (a_k - q + \sigma) + (q + \sigma + \delta)\} + \delta
\]

\[
\rho(s,t) \leq \max \{2\sigma + \delta, 2\sigma + \delta\} + \delta
\]

\[
\rho(s,t) \leq 2(\sigma + \delta)
\]

In the same spirit, with \(\ell(z) = a_n\), assume that \(z = u^{-1}s = v^{-1}t\) with \(\ell(\tilde{s}), \ell(\tilde{t}) \leq a_n - q + \sigma\) and \(\ell(u), \ell(v) \leq q + \sigma + \delta\). We now examine the quadruple \((e,z,s,t) \in G^4:\)

\[
\rho(e,z) + \rho(\tilde{s},\tilde{t}) \leq \max \{\rho(e,\tilde{s}) + \rho(z,\tilde{t}), \rho(e,\tilde{t}) + \rho(z,\tilde{s})\} + \delta
\]

\[
a_n + \rho(\tilde{s},\tilde{t}) \leq \max \{(a_n - q + \sigma) + (q + \sigma + \delta), (a_n = q + \sigma) + (q + \sigma + \delta)\} + \delta
\]

\[
\rho(\tilde{s},\tilde{t}) \leq \max \{2\sigma + \delta, 2\sigma + \delta\} + \delta
\]

\[
\rho(\tilde{s},\tilde{t}) \leq 2(\sigma + \delta)
\]

Thus, for a given \(y \in E_{a_k}\), the number of possible decompositions \(y = \tilde{x}u\) is bounded by the cardinality of the ball of radius \(2(\sigma + \delta)\). Similarly, for a given \(z \in E_{a_n}\), the number of decompositions \(z = \tilde{u}v\) is bounded by the cardinality of the ball of radius \(2(\sigma + \delta)\). Denote by \(N\) the cardinality of the ball of radius \(2(\sigma + \delta)\). Then we proceed:

\[
\sum_{x \in E_{a_m}} |(f \ast \xi)(x)|^2 \leq \sum_{x \in E_{a_m}} \sum_{y \in \mathcal{Y}} \sum_{\ell(u) \leq q + \sigma + \delta} f(\tilde{x}u)\xi(u^{-1}\tilde{x})|^2
\]

\[
\leq \sum_{\ell(u) \leq q + \sigma + \delta} \left(\sum_{\ell(v) \leq q + \sigma + \delta} |f(\tilde{x}u)|^2\right) \left(\sum_{\ell(v) \leq q + \sigma + \delta} |\xi(v\tilde{x})|^2\right)
\]

\[
\leq N^2 \|f\|_2^2 \|\xi\|_2^2
\]

Thus we have a fixed constant, independent of \(m, n,\) and \(k\) such that whenever \(f \subset E_{a_k}\) and \(\xi \subset E_{a_n}\) then

\[
\|P_m \cdot (f \ast \xi)\|_2^2 \leq N^2 \|f\|_2^2 \|\xi\|_2^2
\]

where \(P_m : l^2G \rightarrow l^2E_{a_m}\) is the standard projection. This implies that \(G\) has property RD. \(\square\)
It is not currently known if there exist groups which are pseudohyperbolic, but not hyperbolic.

4 Pseudohyperbolic Groups

**Definition 4.1.** A metric space $(X, d)$ is said to be pseudohyperbolic if there is a positive constant $A$, such that for any four points $w, x, y, z \in X$ we have that:

$$d(w, x) + d(y, z) \leq \max \{d(w, y) + d(x, z), d(w, z) + d(x, y)\} + A \log(1 + \min(w, x, y, z))$$

where $\min(w, x, y, z)$ is the minimum of $\{d(w, x), d(w, y), d(w, z), d(x, y), d(x, z), d(y, z)\}$.

**Example 4.1.** Any hyperbolic space is pseudohyperbolic.

**Remark 4.1.** We can replace $A \log(1 + x)$ by an arbitrary nondecreasing function $F : \mathbb{R}_+ \to \mathbb{R}_+$ and still obtain valid results. For this reason, we will be performing our calculations below using an arbitrary $F$.

**Example 4.2.** Let $\Gamma$ be a polynomial growth group, and let $w, x, y, z \in \Gamma$. Using the triangle inequality we find

$$d(w, x) + d(y, z) \leq (d(w, y) + d(y, x)) + (d(y, w) + d(w, z))$$

$$\leq d(x, y) + d(w, z) + 2d(w, y)$$

$$\leq \max \{d(w, y) + d(x, z), d(w, z) + d(x, y)\} + 2d(w, y)$$

By repeated use of the triangle inequality, we find that

$$d(w, x) + d(y, z) \leq \max \{d(w, y) + d(x, z), d(w, z) + d(x, y)\} + 2d(w, x)$$

$$d(w, x) + d(y, z) \leq \max \{d(w, y) + d(x, z), d(w, z) + d(x, y)\} + 2d(w, z)$$

$$d(w, x) + d(y, z) \leq \max \{d(w, y) + d(x, z), d(w, z) + d(x, y)\} + 2d(x, z)$$

$$d(w, x) + d(y, z) \leq \max \{d(w, y) + d(x, z), d(w, z) + d(x, y)\} + 2d(y, z)$$

Thus we have

$$d(w, x) + d(y, z) \leq \max \{d(w, y) + d(x, z), d(w, z) + d(x, y)\} + 2 \min(w, x, y, z)$$
so that $\Gamma$ satisfies the pseudohyperbolic inequality, with the logarithm replaced by $F(x) = 2x$. As the above computation relies on nothing but the triangle inequality, we see that any metric space can will satisfy the inequality with the logarithm replaced by $F(x) = 2x$.

**Lemma 4.1.** Let $\Gamma$ be a countable group with a uniformly discrete $\sigma$-quasiconvex length function, $\ell$, taking values $\{0, a_1, a_2, \ldots \}$. If $\Gamma$ satisfies the above inequality with function $F$ in the metric induced by $\ell$, then $\Gamma$ has the following property:

$$
\|P_{a_m}(f \ast \xi)\|^2 \leq N(2(\sigma + F(a_k)))N(2\sigma + F(a_k) + F(a_k + F(a_k) + \sigma))\|f\|^2\|\xi\|^2
$$

whenever $f, \xi \in \mathcal{C}\Gamma$ with supp $f \subset E_{a_k}$, and supp $\xi \subset E_{a_n}$, and $P_{a_m} : l^2\Gamma \to l^2E_{a_m}$ is the standard projection, and $N(r)$ is the cardinality of the ball of radius $r$.

**Proof.** Let the pseudohyperbolicity function of $(\Gamma, \ell)$ be denoted by $F$, and quasiconvexity constant $\sigma$. Denote by $d$ the metric induced by $\ell$.

Now, let $f, \xi \in \mathcal{C}\Gamma$ with supp $f \subset E_{a_k}$ and supp $\xi \subset E_{a_n}$, and $x \in E_{a_m}$. Let $q = (a_n + a_k - a_m)/2$. Then:

$$(f \ast \xi)(x) = \sum_{x = yz} f(y)\xi(z) = \sum_{x = yz, y \in E_{a_k}, z \in E_{a_n}} f(y)\xi(z)$$

As $\ell$ is quasiconvex, $D_{a_k}^\sigma(x)$ is nonempty for each $x \in E_{a_m}$. Thus for each $x$ we will fix once and for all, one pair $(\bar{x}, \tilde{x}) \in D_{a_k}^\sigma(x)$. If $x = yz$ as in the sum, examine the quadruple $(e, x, \bar{x}, y) \in \Gamma^4$:

$$
d(e, x) + d(\bar{x}, y) \leq \max \{d(e, \bar{x}) + d(x, y), d(e, y) + d(x, \bar{x})\} + F(\min(e, x, \bar{x}, z))$$

$$
a_m + d(\bar{x}, y) \leq \max \{(a_k - q + \sigma) + a_n, a_k + (a_n - q + \sigma)\} + F(\min(e, x, \bar{x}, z))$$

$$
d(\bar{x}, y) \leq (a_k + a_n - a_m) - q + \sigma + F(\min(e, x, \bar{x}, z))$$

$$
da(\bar{x}, y) \leq q + \sigma + F(\min(e, x, \bar{x}, z))$$

As $d(x, z) = \ell(y) = a_k$ we have the following estimate:

$$
d(\bar{x}, y) \leq q + \sigma + F(a_k)$$
Thus, for $u = \bar{x}^{-1} y$ we have $y = \bar{x} u$ with $\ell(u) \leq q + \sigma + F(a_k)$. This also yields a decomposition $z = u^{-1} \bar{x}$. We then have:

$$(f \ast \xi)(x) = \sum_{u \in G, \ell(u) \leq q + \sigma + F(a_k)} f(\bar{x} u) \xi(u^{-1} \bar{x})$$

For a given $y \in E_{a_k}$, how many factorizations of the form $y = su$ with $\ell(s) \leq a_k - q + \sigma$ and $\ell(u) \leq q + \sigma + F(a_k)$ are there? Assume that $y = su = tv$ with $\ell(s), \ell(t) \leq a_k - q + \sigma$ and $\ell(u), \ell(v) \leq q + \sigma + F(a_k)$. Consider the quadruple $(e, y, s, t) \in G^4$:

$$d(e, y) + d(s, t) \leq \max \{d(e, s) + d(y, t), d(e, t) + d(y, s)\} + F(\min(e, y, s, t))$$
$$a_k + d(s, t) \leq (a_k - q + \sigma) + (q + \sigma + F(a_k)) + F(\min(e, y, s, t))$$
$$d(s, t) \leq 2\sigma + F(a_k) + F(\min(e, y, s, t))$$

Now, $d(e, y) = \ell(y) = a_k$ so that we have

$$d(s, t) \leq 2(\sigma + F(a_k))$$

In the same spirit, with $\ell(z) = a_n$, assume that $z = u^{-1} \bar{s} = v^{-1} \bar{t}$ with $\ell(\bar{s}), \ell(\bar{t}) \leq a_n - q + \sigma$ and $\ell(u), \ell(v) \leq q + \sigma + F(a_k)$. We now examine the quadruple $(e, z, \bar{s}, \bar{t}) \in G^4$:

$$d(e, z) + d(\bar{s}, \bar{t}) \leq \max \{d(e, \bar{s}) + d(z, \bar{t}), d(e, \bar{t}) + d(z, \bar{s})\} + F(\min(e, z, \bar{s}, \bar{t}))$$
$$a_n + d(\bar{s}, \bar{t}) \leq (a_n - q + \sigma) + (q + \sigma + F(a_k)) + F(\min(e, z, \bar{s}, \bar{t}))$$
$$d(\bar{s}, \bar{t}) \leq 2\sigma + F(a_k) + F(\min(e, z, \bar{s}, \bar{t}))$$

Now, $d(\bar{s}, z) = \ell(u^{-1}) \leq q + \sigma + F(a_k) \leq a_k + \sigma + F(a_k)$. Thus,

$$d(\bar{s}, \bar{t}) \leq 2\sigma + F(a_k) + F(a_k + F(a_k) + \sigma)$$

Thus, for a given $y \in E_{a_k}$, the number of possible decompositions $y = \bar{x} u$ is bounded by the cardinality of the ball of radius $2(\sigma + F(a_k))$. Denote by $C_1$ the cardinality of the ball of radius $2(\sigma + F(a_k))$. Similarly, for a given $z \in E_{a_n}$, the number of decompositions $z = u^{-1} \bar{x}$ is bounded by the cardinality of the ball of radius $2\sigma + F(a_k) + F(a_k + F(a_k) + \sigma)$. Denote by $C_2$ the cardinality of the ball of radius $2\sigma + F(a_k) + F(a_k + F(a_k) + \sigma)$. Then we proceed:
\[
\sum_{x \in E_{a_m}} |(f * \xi)(x)|^2 = \sum_{x \in E_{a_m}} \left| \sum_{x=yz} f(y)\xi(z) \right|^2 \\
\leq \sum_{x \in E_{a_m}} \left| \sum_{\ell(u) \leq q + \sigma + F(a_k)} f(\tilde{x}u)\xi(u^{-1}\tilde{x}) \right|^2 \\
\leq \sum_{x \in E_{a_m}} \left( \sum_{\ell(u) \leq q + \sigma + F(a_k)} |f(\tilde{x}u)|^2 \right) \left( \sum_{\ell(v) \leq q + \sigma + F(a_k)} |\xi(v\tilde{x})|^2 \right) \\
\leq (C_1\|f\|_2^2) (C_2\|\xi\|_2^2) \\
\leq C_1 C_2 \|f\|_2^2 \|\xi\|_2^2
\]

Corollary 4.1. Let $\Gamma$ be a group endowed with a uniformly discrete, $\sigma$-quasiconvex length function $\ell$, such that $(\Gamma, \ell)$ is of polynomial growth. Then $\Gamma$ has property RD with the length $\ell$.

Proof. As $\Gamma$ is of polynomial growth, there is a polynomial $Q$ such that the cardinality of $B^\ell_r$ is bounded by $Q(r)$, and $\Gamma$ is pseudo-hyperbolic with $F(x) = 2x$. Let $f, \xi \in \mathbb{C}^\Gamma$ with $\text{supp } f \subset E_{a_k}$, and $\text{supp } \xi \subset E_{a_n}$. Then from Lemma 4.1, we have that

$$\|P_{a_m} (f * \xi)^2 \leq Q(2\sigma + 4a_k))Q(4\sigma + 8a_k)\|f\|_2^2 \|\xi\|_2^2$$

Using the fact that $Q(2\sigma + 4a_k)Q(4\sigma + 8a_k)$ is a polynomial in $a_k$, the Corollary follows. \qed

Corollary 4.2. Let $\Gamma$ be a finitely generated group with algebraic word length function $|\cdot |$, such that $(\Gamma, | \cdot |)$ is pseudo-hyperbolic with the function $F(x) = A\log(1 + x)$. Then $\Gamma$ has property RD with the word length function.

Proof. Let $M$ be the cardinality of the symmetric finite generating set for $\Gamma$, yielding $| \cdot |$. We can assume that the base of the logarithm appearing in $F$ is $M$. Then the cardinality of the ball of radius $2\sigma + A\log(1 + a_k)$ is bounded by $M^{2\sigma + A\log(1 + a_k)} = M^{2\sigma}(1 + a_k)^A$. Similarly the cardinality of the ball of radius $2\sigma + A\log(a_k) + A\log(a_k + A\log(a_k) + \sigma)$ is bounded by $M^{2\sigma + A\log(1 + a_k) + A\log(1 + a_k + A\log(1 + a_k) + \sigma)} = M^{2\sigma}(1 + a_k)^A(1 + a_k + A\log(1 + a_k) + \sigma)$. Therefore, the Corollary holds. \qed
\( \sigma \). These are both bounded by polynomials in \( a_k \), so applying Lemma 4.1, we have that
\[
\|P_{\alpha_m}(f \ast \xi)\|_2^2 \leq Q(a_k)\|f\|_2^2\|\xi\|_2^2
\]
where \( f \) and \( \xi \) are as above, and \( Q \) is a polynomial. The Corollary follows. \( \square \)

5 Relatively Pseudohyperbolic Groups

In [CR] Chatterji and Raune show that groups which are hyperbolic relative to a family of polynomial growth subgroups have property RD. Druțu and Sapir generalized this result to \((\ast)\)-relatively hyperbolic groups, and note that all strongly hyperbolic groups are \((\ast)\)-relatively hyperbolic [DS].

Druțu and Sapir’s concept of \((\ast)\)-relative hyperbolicity can be generalized in the same spirit as above.

Let \( \Gamma \) be a finitely generated discrete group with a symmetric finite generating set \( S \) of cardinality \( M \), let \( \cdot \) be the corresponding word length function, and let \( \rho(x,y) = |x^{-1}y| \). Let \( H_1, H_2, \ldots, H_j \) be subgroups of \( \Gamma \). We say that \( \Gamma \) is \textit{Relatively Pseudohyperbolic with respect to} \( \{ H_1, H_2, \ldots, H_j\} \) if there exists positive constants \( C \) and \( N \) such that polynomial \( Q(x) = C(1 + x)^N \) satisfies the following property:

For every geodesic triangle \( ABC \) in the Cayley graph of \( \Gamma \) with respect to \( S \), there exists a coset \( \gamma H_i \) such that the neighborhood \( \mathcal{N}^\log_M Q(\min(A, B, C)) (\gamma H_i) \) intersects each of the sides of the triangle, where \( \min(A, B, C) \) is the minimum side length of the triangle \( ABC \). In addition, the entry and exit points \( A_1, B_1, C_1, \) and \( A_2, B_2, C_2, \) of the sides \([A, B], [B, C], [C, A]\) into the neighborhood \( \mathcal{N}^\log_M Q(\min(A, B, C)) (\gamma H_i) \) satisfy
\[
\rho(A_1, A_2) \leq \log_M Q(\min(A, B, C))
\]
\[
\rho(B_1, B_2) \leq \log_M Q(\min(A, B, C))
\]
\[
\rho(C_1, C_2) \leq \log_M Q(\min(A, B, C))
\]

In what follows, let \( \delta(x) = \log_M Q(x) \).

For every \( x \in \Gamma \) with \( |x| = r \), fix some geodesic segment, \( q_x \), connecting \( x \) to \( e \), the identity element of the group. We thus obtain a set of geodesics, each
of length $r$, corresponding to the elements of $E_r$. Denote this set of geodesics by $G(r)$ and identify it with the set $E_r$ in the obvious way.

Let $q_x$ the a geodesic in $G(m)$, and consider any geodesic triangle with $q_x$ as one of the sides. Call the other sides $q_y$ and $q_z$. Assume that $q_y \in G(k)$ and $q_z \in G(n)$, and let $\min = \min(m, k, n)$ be the minimum of the side lengths. A choice of such a triangle corresponds to a factorization $x = yz$.

Apply the relative pseudohyperbolic property to this triangle, we obtain a left coset $\gamma H_i$ of one of subgroups $\{H_1, H_2, ..., H_j\}$, such that the neighborhood $\tilde{\mathcal{N}}_{\min}(\gamma H_i)$ intersects each of the sides of this triangle. Label the entry and exit points of the edges into the neighborhood by $A_1$, $B_1$, $C_1$, and $A_2$, $B_2$, $C_2$.

As $A_1$ is in the neighborhood, there is a point $x_1$ in the coset $\gamma H_i$, with $\rho(x_1, A_1) \leq \delta(\min)$. Similarly, there is a point $x_1\eta \in \gamma H_i$ with $\rho(x_1\eta, C_1) \leq \delta(\min)$, and a point $x_1\eta' \in \gamma H_i$ with $\rho(x_1\eta', B_1) \leq \delta(\min)$. This choice of $x_1$ and $x_1\eta$ corresponds to factorizations $x = x_1\eta x_2$, $y = x_1\eta' x_3$, and $z = x_2^{-1}\eta'' x_3$ where $x_2 = \eta^{-1}x_1^{-1}x$, $x_3 = (\eta')^{-1}x_1^{-1}y$, and $\eta'' = (\eta')^{-1}\eta$.

From the relative pseudohyperbolic property above, we have that $\rho(A_1, A_2) \leq \delta(\min)$, $\rho(B_1, B_2) \leq \delta(\min)$, $\rho(C_1, C_2) \leq \delta(\min)$. Thus, $\rho(x_1, A_2) \leq 2\delta(\min)$, $\rho(x_1, B_2) \leq 2\delta(\min)$, and $\rho(x_1, C_2) \leq 2\delta(\min)$.

Denote by $\Delta$ the set of all 6-tuples $(x_1, x_2, x_3, \eta, \eta', \eta'') \in \Gamma^3 \times \bigsqcup_{i=1}^4 (H_i)^3$ such that:
(1) \( x = x_1 \eta x_2 \) with \( |x| = m \), \( y = x_1 \eta' x_3 \) with \( |y| = k \) and \( z = x_3^{-1} \eta'' x_2 \) with \( |z| = n \)

(2) \( \eta = \eta' \eta'' \)

(3) If \((\eta, \eta', \eta'') \in (H_i)^3 \) for some \( i \in \{1, 2, \ldots, j\} \), then the following hold, with \( \text{min} = \text{min}\{m, k, n\} \) as above:

- \( x_1 \eta \) is of distance at most \( 2\delta(\text{min}) \) from the exit point \( A_2 \) of \( q_x \) from \( \mathcal{N}_{\delta(\text{min})}(x_1 H_i) \) and from the entrance point \( A_1 \) of \( q_y \) into \( \mathcal{N}_{\delta(\text{min})}(x_1 H_i) \)

- \( x_1 \eta' \) is of distance at most \( 2\delta(\text{min}) \) from the entrance point \( C_1 \) of \( q_x \) into \( \mathcal{N}_{\delta(\text{min})}(x_1 H_i) \) and from the exit point \( C_2 \) of \( yq_x \) from \( \mathcal{N}_{\delta(\text{min})}(x_1 H_i) \)

- \( x_1 \eta'' \) is of distance at most \( 2\delta(\text{min}) \) from the exit point \( B_2 \) of \( q_y \) from \( \mathcal{N}_{\delta(\text{min})}(x_1 H_i) \) and from the entrance point \( B_1 \) of \( yq_x \) into \( \mathcal{N}_{\delta(\text{min})}(x_1 H_i) \).

The set \( \Delta \) is called the set of central decompositions of geodesic triangles with edges in \( \mathcal{G}(m) \times \mathcal{G}(k) \times \mathcal{G}(n) \). The construction above shows that this set is nonempty.

Given an \( x \) of length \( r \), with \( r \in \{m, n\} \), we call every decomposition \( x = x_1 \eta x_2 \) of \( x \) corresponding to a central decomposition in \( \Delta \) a central decomposition of \( x \).

**Theorem 5.1.** Let \( \Gamma \) be a group which is relatively pseudohyperbolic with respect to the subgroups \( \{H_1, H_2, \ldots, H_j\} \). Then \( \Gamma \) has the RD property if and only if \( \{H_1, H_2, \ldots, H_j\} \) have the RD property with respect to the length function induced by a word length function on \( \Gamma \).

As in the case of (*)-Relative Hyperbolicity, the heart of the proof is the following lemma.

**Lemma 5.1.** There exists a polynomial \( F \) such that for any \( w \in E_r \), with \( r \in \{m, n\} \), the number of central decompositions of \( w \) is bounded by \( F(k) \).

**Proof.** Suppose that \( w_1 \) is the left part of some central decomposition \((w_1, \eta, w_2)\) of \( w \). Then \( w_1 \) is at distance at most \( 2\delta(\text{min}) \) from a point lying on some geodesic of length \( k \), coming from some geodesic triangle, \( T \). Thus \(|w_1| \leq \)

13
As one side of $T$ has length $k$ we have that $\min \leq k$, thus $|w_1| \leq k + 2\delta(k)$. Similarly $w_1$ is at distance at most $2\delta(k)$ from a vertex $w'_1$ on $q_w$, so that $|w'_1| \leq k + 4\delta(k)$. As $q_w$ is a geodesic line segment, and $w'_1$ must lie on this segment, there are at most $k + 4\delta(k)$ possibilities for $w'_1$. Fix one such a $w'_1$.

For each $i$, the number of left cosets $\gamma H_i$ at distance at most $2\delta(k)$ from $w'_1$ is bounded by the cardinality of the $2\delta(k)$ neighborhood of $w'_1$. Thus the number of such cosets is bounded by $M^{2\delta(k)} = Q(k)^2$. The total number of cosets $\gamma H_i$, for any $i$, within this distance is bounded by $j \cdot Q(k)^2$. Therefore the number of left cosets $\gamma H_i$ for which $q_w$ has entrance point $w'_1$ is bounded by a polynomial in $k$, independant of $w$.

The exit point, $w'_2$, of $q_w$ from $\mathcal{N}_d(k)(\gamma H_i)$ is uniquely determined each time the left coset is fixed, as it is the point in the intersection of $q_w$ and $\mathcal{N}_d(k)(\gamma H_i)$ that is closest to $w$.

For each left coset, the number of points in that coset at a distance no more than $2\delta(k)$ from $w'_1$ is bounded by the cardinality of the ball of radius $2\delta(k)$ around $w'_1$. The number of such points is, therefore, bounded by $M^{2\delta(k)} = Q(k)^2$, and likewise for $w'_2$.

Thus, given a fixed $w$, the number of possibilities for $w'_1$ is bounded by $k + 4\delta(k)$. For each choice of $w'_1$, there are $j \cdot Q(k)^2$ cosets $\gamma H_i$ for which $w'_1$ can be the entrance point of $q_w$ into $\mathcal{N}_d(k)(\gamma H_i)$. Then, once the coset is fixed, there are at most $Q(k)^2$ possible choices for $w_1$, and $Q(k)^2$ possible choices for $w_1\eta$. $w_2$ is uniquely determined by $w$, $w_1$, and $w_1\eta$. There is some polynomial $F$ such that $(k + 4\delta(k)) (j \cdot Q(k)^2) (Q(k)^2) (Q(k)) \leq F(k)$. Thus the total number of possible central decompositions $(w_1, \eta, w_2)$ for $w$ is bounded by $F(k)$. Moreover, we see that this polynomial is independant of $w$. 

With this lemma, the proof of the theorem follows as in Sapir and Druțu. While the theorem presented here is more general than that of the (*)-Relative Hyperbolic version, they define the notion of (**) -Relative Hyperbolicity which may be more general than what is presented here. We feel that the benefit of this notion, is that it is more geometric than the abstract property of (**) -relative hyperbolicity.
References


Ronghui Ji
Department of Mathematical Sciences
IUPUI
Indianapolis, IN 46202
ronji@math.iupui.edu

Bobby Ramsey
Department of Mathematical Sciences
IUPUI
Indianapolis, IN 46202
bramsey@math.iupui.edu