Composition of Parametrizations,
Using the Paired Algebras of Forms and Sites
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Abstract

An elegant mathematical setting for Bézier curves and surfaces, proposed by Lyle Ramshaw, consists of two copies of a polynomial algebra and a pairing between them. One copy is used for the component functions of the parametrization. The other, in dimension one, represents the domain points. Elements of higher degree in the second algebra are called sites and are useful for explaining blossoming and other constructions performed on Bézier curves and surfaces. This paper extends these definitions to mixed polynomials in two sets of variables, to give an elegant description of the composition of two Bézier parametrizations. In case the first transformation is linear, the construction is related to a classical group representation.

Keywords

Bézier curves, Bézier surfaces, Bézier simplices, blossoming, composition, sites, mixed polynomials.
1. Introduction

There are many situations in computer aided geometric design in which we must compose two parametrizations. One is to describe trim curves on a surface. Suppose $f$ parametrizes a curve in the plane, and that this same plane is the domain of a parametrization $g$ of a surface. Then the image of $g \circ f$ is a curve that lies in the surface. If $f$ is a Bézier curve determined by its control points $B_\alpha$, and $g$ is a Bézier surface determined by its control points $C_\beta$, then $g \circ f$ is also a Bézier curve and thus is determined by control points $A_\gamma$, which somehow depend on the $B_\alpha$ and the $C_\beta$. One object of this paper is to give a formula for the $A_\gamma$.

The first paper on this subject was by DeRose (1988). It was reworked in (DeRose et al, 1993) using the new concept of blossoming that had just been introduced by Ramshaw (1987, 1989). Some effort was required of the reader to learn the machinery of polar forms, but then the composition problem was answered in a more elegantly conceptual way. They also give a number of other examples of compositions in computer graphics and CAGD.

More recently Ramshaw has introduced another algebraic technique, the paired algebras of forms and sites (2001). This language is the natural language to describe (or perhaps replace) the blossoming concept. While again there is some overhead to learn background algebra, the payoff is an extremely simple formula for describing composition; see Theorem 9.

The genesis of this paper was a question from Jorge Estrada: If $g$ is a homogeneous polynomial and you make a linear substitution for the variables, what is an efficient way to compute the coefficients of the new polynomial? This problem arises whenever $g$ is one of the component functions of a Bézier curve or surface and a linear transformation is applied to the domain space.

Obviously this is another example of the composition of two parametrizations. In this case, it also has an answer given by classical representation theory. The representation theory construction uses polynomial multiplication as a tool, which is what suggested the use of Ramshaw’s paired algebras. This paper includes a comparison of these two solution methods.

The organization of the paper is as follows. First we review the paired algebras of forms and sites and then formulas for the evaluation of a form at a point. Then we introduce mixed polynomials and compositions of these. We give an example of a trim curve on a cubic surface patch. Finally we compare two solution methods for the case of
a linear transformation of coordinates in the domain.

Lin and Walker are also developing the ideas of Ramshaw; see (Lin et al, 2004, 2005) and (Walker, 2004).

2. The Paired Algebras of Forms and Sites

Let \( \mathcal{H}_*(R^r) \) be the real polynomial algebra in \( r \) variables \( x_1, x_2, \ldots, x_r \). This algebra, also called a \textit{symmetric algebra}, is graded by degree. That is, every polynomial is the sum of homogeneous polynomials of degree \( n, n \geq 0 \). Thus \( \mathcal{H}_*(R^r) \) is the direct sum of the vector subspaces \( \mathcal{H}_n(R^r) \); the homogeneous polynomials, or \textit{forms}, of degree \( n \) in \( r \) variables. While each subspace \( \mathcal{H}_n(R^r) \) is closed with respect to addition and scalar multiplication, the product of an element of \( \mathcal{H}_m(R^r) \) and an element of \( \mathcal{H}_n(R^r) \) is in \( \mathcal{H}_{m+n}(R^r) \). This makes \( \mathcal{H}_*(R^r) \) a \textit{graded algebra}.

The \textit{monomial basis} of \( \mathcal{H}_n(R^r) \) consists of all \( x^\alpha := x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_r^{\alpha_r} \) for multi-indices \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \) of \( r \) non-negative integers that sum to \( n \), written \( |\alpha| = n \). For technical reasons, however, we use the basis \( \{\binom{n}{\alpha}x^\alpha\} \) for forms. The dimension of \( \mathcal{H}_n(R^r) \) is \( \binom{n+r-1}{n} \), with a typical element \( f(x_1, \ldots, x_r) = \sum_\alpha \binom{n}{\alpha} f_\alpha x^\alpha \).

The \textit{algebra of sites} is a second copy of the symmetric algebra, denoted \( \mathcal{S}_*(R^r) \). To distinguish sites from forms we use \( X_1, \ldots, X_r \) as the basis elements for sites. We use the monomial basis, without binomial coefficients, for sites. In dimension one, \( \mathcal{S}_1(R^r) \) is isomorphic as a vector space to \( R^r \) by letting the point \( P(p_1, \ldots, p_r) \) correspond to the site \( p(X_1, \ldots, X_r) = p_1X_1 + \cdots + p_rX_r \). We will use this isomorphism to identify the 1-sites \( \mathcal{S}_1(R^r) \) with points. (Note however that this notation conflicts with that set up in the previous paragraph, in that we are using \( p_1 \) for \( p_{(1,0,\ldots,0)} \), for example.) In degree 0, both \( \mathcal{H}_0(R^r) \) and \( \mathcal{S}_0(R^r) \) are naturally isomorphic to \( R \).

There are several advantages to writing the point \( P(p_1, \ldots, p_r) \) as \( p = \sum_i p_iX_i \). The first is that we now have the ability to multiply points. The product of two points is a 2-site. In general the product of an \( m \)-site \( p \) and an \( n \)-site \( q \) is an \((m+n)\)-site \( pq \). The \( n \)-th power of the point \( P \) is the \( n \)-site \( p^n \). It is this ability to multiply points that is used to give an elegant definition of blossoming — or perhaps to dispense with the term blossoming altogether. See (Ramshaw, 2001).

A second advantage is that the terms of the site \( p = \sum_i p_iX_i \) can be written in any order, whereas the components of the vector \( P(p_1, \ldots, p_r) \) have a fixed order. This becomes important for sites of higher degree when the number of variables \( r \) is greater than 2. The monomials in the \( X_i \) now distinguish the terms.
We now review Ramshaw’s definition of the pairing between $n$-forms and $n$-sites. In degree one it is defined by $\langle X_i, x_j \rangle = \delta_{ij}$ for basis elements and then extended linearly. Thus in degree 1 it is the standard dot product.

In degree $n$ the pairing is first defined for lineal $n$-forms. An $n$-form or $n$-site is lineal if it is the product of 1-forms or 1-sites. After the pairing is defined for lineal $n$-forms and $n$-sites it is then extended linearly, using the following Proposition.

**Proposition 1.** Every $n$-form can be written as the sum of lineal $n$-forms and also as the sum of perfect $n$th powers. The same is true for sites. This is proved in (Ramshaw, 2001), Lemmas 7.2–1 and 7.2–2.

The pairing is defined for lineal $n$-forms and $n$-sites by using the **Summed Permanent Identity** for $1$-sites $p_1, \ldots, p_n$ and $1$-forms $f_1, \ldots, f_n$,

$$\langle p_1 \cdots p_n, f_1 \cdots f_n \rangle = \sum_{\nu \in S_n} \prod_{k \in [1..n]} \langle p_{\nu(k)}, f_k \rangle,$$

where the summation index $\nu$ varies over the symmetric group $S_n$ of all $n!$ permutations of the integers from 1 to $n$.

Here are some of properties of the pairing. These are derived directly from the definitions and are proved in (Ramshaw, 2001).

**Proposition 2.**

(i) For $a$ in $S_0(R^n)$ and $b$ in $H_0(R^n)$, $\langle a, b \rangle = ab$.

(ii) In each dimension $n$, the pairing is nonsingular and makes each of $S_n(R^n)$ and $H_n(R^n)$ the dual vector space of the other.

(iii) In the Summed Permanent Identity, if all the $p_i = p$ and all the $f_i = f$, then

$$\langle p^n, f^n \rangle = n! \langle p, f \rangle^n.$$

(iv) If $X^\alpha$ and $x^\beta$ have degree $n$,

$$\langle X^\alpha, x^\beta \rangle = \begin{cases} 0, & \text{if } \alpha \neq \beta \\ \alpha!, & \text{if } \alpha = \beta \end{cases}$$

where $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_s!$.

The pairing of $S_n(R^n)$ and $H_n(R^n)$, because it is bilinear, can also be described in matrix notation where it is given in terms of a symmetric matrix $Q_n$. 

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Proposition 3. Let $S$ be the row matrix of components of an $n$-site $s$. Let $F$ be the column matrix of coefficients of an $n$-form $f$, omitting the binomial coefficients. Then

$$\langle s, f \rangle = SQ_nF = n!SF$$

Proof. In dimension one, $Q_1$ is the identity $r \times r$ matrix. However as a consequence of Proposition 2(iv), the matrix $Q_n$ of the pairing in degree $n > 1$ is diagonal, but is $n!$ times the identity matrix because the element $Q_{n\alpha,\alpha} = \langle X^{\alpha}, (^{n}_\alpha)x^{\alpha} \rangle = n!$.

As an illustration, if $r = n = 2$, $s = s_{2,0}X_1^2 + s_{1,1}X_1X_2 + s_{0,2}X_2^2$ and $f = f_{2,0}x_1^2 + 2f_{1,1}x_1x_2 + f_{0,2}x_2^2$,

$$\langle s, f \rangle = \begin{pmatrix} s_{2,0} & s_{1,1} & s_{0,2} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} f_{2,0} \\ f_{1,1} \\ f_{0,2} \end{pmatrix} = 2(s_{2,0}f_{2,0} + s_{1,1}f_{1,1} + s_{0,2}f_{0,2})$$

Of course writing a vector or a matrix requires selecting an ordering for the basis. This is easy enough in case $r = 2$, but requires an arbitrary choice when $r > 2$ and the space has dimension $\binom{n+r-1}{n}$. The example above uses lexicographical order. Whichever ordering we select, we must use it for row vectors, column vectors and matrices. This shows another advantage of the forms and sites notation: the subscripts on the $x_i$ and $X_i$ replace having to keep track of the ordering.

3. Evaluation

There are three notations for the evaluation of an $n$-form $f$ at $P$. If $P$ is a point and $p$ the corresponding 1-site, we reuse the letter $P$ for the row vector of its coordinates and write $P^n$ for the row vector of the $\binom{n+r-1}{n}$ components of $p^n$, using the order that has been chosen. We write the coefficients of $f$, omitting the binomial coefficients, as a column vector $F$.

Proposition 4. The following expressions for the evaluation of an $n$-form $f$ at $P$ are equivalent:

(i) the usual evaluation notation $f(P)$,
(ii) pairing notation $\langle p^n/n!, f \rangle$,
(iii) matrix notation $P^nF$.

Proof. The proof of (ii) is given in (Ramshaw, 2001), Prop. 7.6-1. We state a proof here in such a way that it can be reused later for Proposition 7. Note $f(P)$ is obtained by
substituting the coefficients of the $X_i$ in $p$ for the $x_i$ in $f$, thus the term $f_\alpha \binom{n}{\alpha} x^\alpha$ becomes $f_\alpha \binom{n}{\alpha} p^\alpha$. Now in the pairing $\langle p^n / n!, f \rangle$, the only term of $p^n / n!$ that pairs nonzero with $f_\alpha \binom{n}{\alpha} x^\alpha$ is $\binom{n}{\alpha} \frac{1}{\alpha!} p^\alpha X^\alpha$, and the result is $f_\alpha \binom{n}{\alpha} p^\alpha$. Part (iii) follows from Proposition 3.

4. Mixed polynomials

In this section the new algebra needed to describe the composition of polynomial mappings is introduced.

We now consider a second vector space $R^s$. Let $y_i$ and $Y_i$ be the variables of its pair of algebras $\mathcal{H}_s(R^s)$ and $\mathcal{S}_s(R^s)$.

A mixed polynomial is a polynomial in two sets of variables and is analogous to a function from $R^r$ to $R^s$.

Define $\mathcal{H}_{k,l}(r, s)$ to be the set of polynomials that are homogeneous of degree $k$ in $x_1, \ldots, x_r$ and homogeneous of degree $l$ in $Y_1, \ldots, Y_s$. A typical element of $\mathcal{H}_{k,l}(r, s)$ is written

$$\mathcal{F} = \sum_{|\alpha| = k, |\beta| = l} a_{\alpha, \beta} x^\alpha Y^\beta$$

The set of all mixed polynomials $\mathcal{H}_{r,s}(r, s)$ is an algebra, bigraded by $(k, l)$.

There are two important special cases. If $k = 0$ there are no variables $x$ in the polynomial. The $r$ becomes irrelevant and $\mathcal{H}_{0,l}(r, s)$ reduces to $\mathcal{S}_l(R^s)$. If $l = 0$ there are no variables $Y$, the $s$ becomes irrelevant, the binomial coefficients remain, and $\mathcal{H}_{k,0}(r, s)$ reduces to $\mathcal{H}_k(R^r)$. Thus both sites and forms are special cases.

Now consider yet a third space $R^t$, with its two sets of variables $z_i$ and $Z_i$.

We use the pairing to define the operation of composition

$$\mathcal{H}_{k,l}(r, s) \otimes \mathcal{H}_{l,m}(s, t) \to \mathcal{H}_{k,m}(r, t)$$

by letting

$$\mathcal{F} \circ \mathcal{G} = \langle \mathcal{F} / l! \rangle \cdot \mathcal{G}$$

The mixed polynomial on the left is treated as a site, while that on the right is treated as a form. The $Y$s in $\mathcal{F}$ pair up with the $y$s in $\mathcal{G}$, leaving a mixed polynomial in the $x$s and the $Z$s.

If $\mathcal{G}$ is the mixed polynomial $\sum \binom{l}{m} b_{\gamma, \delta} Y^\gamma Z^\delta$, the expanded form for this composition is

$$\sum \binom{l}{m} b_{\gamma, \delta} Y^\gamma Z^\delta$$
\[
\left\{ \sum_{|\alpha|=k} \frac{k}{\alpha} a_{\alpha,\beta} x^\alpha Y^{\beta} / l!, \sum_{|\gamma|=t} \frac{l}{\gamma} b_{\gamma,\delta} y^\gamma Z^\delta \right\} = \\
\sum_{|\alpha|=k} \frac{k}{\alpha} \sum_{|\beta|=l} \frac{l}{\beta} a_{\alpha,\beta} b_{\beta,\delta} (Y^{\beta} / l!, y^\beta) x^\alpha Z^\delta = \\
\sum_{|\alpha|=k} \frac{k}{\alpha} \left( \sum_{|\beta|=l} a_{\alpha,\beta} b_{\beta,\delta} \right) x^\alpha Z^\delta
\]

We will see applications of this composition in the following sections.

5. Polynomial mappings

In this section we see how Bézier parametrizations give rise to mixed polynomials. We apply the composition in this case to find a compact formula for the control points of a composition.

Let \( f : R^r \rightarrow R^s \) be the parametrization of a curve, surface or higher dimensional simplex in \( R^s \) or \( RP^{s-1} \), defined by its control points \( B_\alpha(B_{\alpha,1}, \ldots, B_{\alpha,s}) \) in \( R^s \).

Each of the control points \( B_\alpha \) determines a 1-site \( b_\alpha \) in the range space of \( F \): \( b_\alpha = B_{\alpha,1} Y_1 + \cdots + B_{\alpha,s} Y_s \).

If we choose a basis ordering we can form the \( \binom{n+r-1}{n} \times s \) matrix \( B \) whose rows are the coordinates of the control points.

The \( s \) component functions of \( f \) are \( f_i = \sum_\alpha \binom{n}{\alpha} B_{\alpha,1} x^\alpha \). The coefficients of the \( f_i \) come from the columns of the matrix \( B \).

The parametrization \( f \) determines the mixed polynomial \( \mathcal{F} \in \mathcal{H}_{n,1}(r,s) \),

\[
\mathcal{F} = \sum_\alpha \binom{n}{\alpha} b_\alpha x^\alpha = \sum_i f_i Y_i = \sum_\alpha \sum_i \binom{m}{\alpha} B_{\alpha,i} x^\alpha Y_i
\]

Analogous to Proposition 4, we can express the evaluation of \( f \) both in terms of the pairing and in terms of matrices.

**Proposition 5.** The following expressions for the evaluation of a homogeneous polynomial transformation \( f : R^r \rightarrow R^s \) of degree \( n \) are equivalent, and yield a 1-site in \( R^s \):

(i) the usual evaluation notation \( f(P) \),

(ii) pairing notation \( \langle p^n / n!, \mathcal{F} \rangle \),

(iii) matrix notation \( P^n B \).
Proof. Using Proposition 4(ii),
\[
\langle p^n/n!, \mathcal{F} \rangle = \langle p^n/n!, \sum_i f_i Y_i \rangle = \sum_i \langle p^n/n!, f_i \rangle Y_i = \sum_i f_i(P)Y_i,
\]
which is the 1-site that corresponds to \( f(P) \). 

In matrix notation the Bézier parametrization of a cubic curve, for example, is
\[
((1 - t)^3 \quad 3t(1 - t)^2 \quad 3t^2(1 - t) \quad t^3) B
\]
This is an example of (iii) with the choice of the domain point \( P \) in the unit interval, written in barycentric coordinates \( P(1-t,t) \).

While the matrix formula here looks as simple as the pairing formula, remember that expanding the matrix \( B \) requires careful attention to the ordering of \( \binom{n+r-1}{n} \) basis elements, whereas the terms of \( \mathcal{F} \) can be written in any order.

We can recover the control points of \( f \) by pairing \( \mathcal{F} \) with certain specific domain points.

**Proposition 6.** The components of the control point \( B_\alpha \) are the coefficients of \( b_\alpha \), and
\[
b_\alpha = \langle X^\alpha/n!, \mathcal{F} \rangle
\]
Proof. \( \langle X^\alpha/n!, \mathcal{F} \rangle = \langle X^\alpha/n!, \sum_\alpha \binom{n}{\alpha} b_\alpha x^\alpha \rangle = \binom{n}{\alpha} b_\alpha \langle X^\alpha, x^\alpha \rangle = b_\alpha \).

Now suppose we have a second Bézier parametrization \( g : R^s \to R^t \), of degree \( m \), with control points \( C_\beta \). Here \( \beta \) runs over nonnegative partitions of \( m \) of length \( s \). Associated to \( g \) is the mixed polynomial
\[
G = \sum_\beta \sum_j \binom{m}{\beta} C_{\beta,j} y^\beta Z_j
\]
In order to describe the composition \( g \circ f \) and \( g(f(P)) \) in the language of mixed polynomials, we need the following result that relates pairing and powers.

**Proposition 7.** If \( f \) is a homogeneous polynomial mapping of degree \( n \) and \( p \) is either a 1-site or a mixed polynomial that is degree 1 in the \( \{ X_i \} \), \( \langle p^n/n!, \mathcal{F} \rangle = \langle p^{mn}/(mn)!, \mathcal{F}^m \rangle \).
Proof. As in the proof of Proposition 4, \( \langle p^n/n!, \mathcal{F} \rangle \) is obtained by substituting the coefficients of the \( X_i \) in \( p \) for the \( x_i \) in \( \mathcal{F} \). These coefficients of the \( X_i \) can be constants
or polynomials in yet another set of variables. After substitution, the result is raised to the $m$th power for $(p^n/n!, F)^m$. For the case of $(p^{mn}/(mn)!, F^m)$, $F$ is raised to the $m$th power before substitution. The result is the same.

We introduce the notation $\phi(f) = F, \phi(g) = G$ to have notation for the mixed polynomial $\phi(g \circ f)$ that arises from the composition of $f$ and $g$. This composition can be described in terms of the pairing, with a formula that is the same as that used for evaluation.

Proposition 8. If $f : R^r \to R^s$ is a homogeneous polynomial mapping of degree $n$ and $g : R^s \to R^t$ is a homogeneous polynomial mapping of degree $m$, then

$$\phi(g \circ f) = (\phi(f)^m/m!, \phi(g)) = (F^m/m!, G) = F^m \circ G$$

Proof. Note that $g \circ f$ and $\phi(g \circ f)$ are obtained by substituting the coefficients of the $Y_i$ in $F$ for the $y_i$ in $g$ or $G$. Formally the proof is the same as that of Proposition 4(ii) except that these coefficients are no longer constants.

We can now state the result concerning the control points of a composition.

Theorem 9. Let $f$ be a Bézier simplex of degree $n$ and control points $B_\alpha$ in $R^n$, and $g$ a Bézier simplex of degree $m$ and control points $C_\beta$ in $R^t$. Let $\gamma$ range over the nonnegative partitions of $mn$. Then the control points of $g \circ f$ in $R^t$ are given by

$$A_\gamma = (X^\gamma/(mn)!, \phi(g \circ f)).$$

Proof. In Proposition 6 we observed we could recover the control points of $f$ from $b_\alpha = (X^\alpha/n!, F)$. This is the same principle.

6. Example

Here is an example of closed curve of degree five in the domain of a cubic surface patch, which leads to a curve of degree 15 on the surface.

The Bézier curve in the domain of a triangular surface patch is defined by its control points in barycentric coordinates: $B_{5,0}(\frac{1}{5}, \frac{3}{5}, \frac{1}{5}), B_{4,1}(0, \frac{3}{4}, \frac{1}{4}), B_{3,2}(\frac{1}{4}, 0, \frac{3}{4}), B_{2,3}(\frac{3}{4}, 0, \frac{1}{4}), B_{1,4}(\frac{1}{4}, \frac{3}{4}, 0)$, and $B_{0,5} = B_{(5,0)}$; see Figure 1.
From the control points we obtain the mixed polynomial

$$\mathcal{F} = \frac{1}{8} x_2^5 + \frac{15}{2} x_1^2 x_2^3 + \frac{5}{2} x_1 x_2^4 + \frac{5}{4} x_1^2 x_2^4 + \frac{1}{8} x_1^5 Y_1 + \left( \frac{3}{4} x_2^5 + \frac{15}{4} x_1 x_2^4 + \frac{15}{4} x_1^2 x_2^4 + \frac{3}{4} x_1^5 \right) Y_2 + \\
\left( \frac{1}{8} x_2^5 + \frac{5}{4} x_1^2 x_2^3 + \frac{5}{2} x_1^2 x_2^4 + \frac{15}{2} x_1^3 x_2^3 + \frac{1}{8} x_1^5 \right) Y_3$$

The surface on which this curve is to lie has control points in affine coordinates: $C_{0,0,3}(2,0,0,1), C_{0,1,2}(1,1,1,1), C_{1,0,2}(1,0,1,1), C_{0,2,1}(0,2,0,1), C_{1,1,1}(0,1,1,1), C_{2,0,1}(0,0,1,1), C_{0,3,0}(1,3,2,1), C_{2,1,0}(-1,-1,5,1), C_{1,2,0}(0,1,5,1)$, and $C_{3,0,0}(-2,-3,2,1)$. From these we obtain the mixed polynomial

$$\mathcal{G} = (2 y_3^3 + 3 y_2 y_3^2 + y_2^3 + 3 y_1 y_3^2 - 3 y_1^2 y_2 - 2 y_1^3) Z_1 + \\
(3 y_2 y_3^2 + 6 y_2 y_3 + 3 y_2^3 + 6 y_1 y_3 y_2 + 3 y_1 y_2^3 - 3 y_1 y_2 - 3 y_1^3) Z_2 + \\
(3 y_2 y_3^2 + 2 y_2^3 + 3 y_1 y_3^2 + 6 y_1 y_2 y_3 + 15 y_1 y_2^3 + 15 y_1 y_2 + 2 y_1^3) Z_3 + \\
(y_3^3 + 3 y_2 y_3^2 + 3 y_2 y_3 + y_2^3 + 3 y_1 y_3^2 + 6 y_1 y_2 y_3 + 3 y_1 y_2^3 + 3 y_1^2 y_2 + y_1^3) Z_4$$

Now let $\mathcal{H} = \langle \mathcal{F}^3/3!, \mathcal{G} \rangle$. The control points of $\mathcal{H}$ are $A_{(15-i,i)} = \langle X_1^{15-i} X_2^i/15!, \mathcal{H} \rangle$:

$$A_{15,0}(\frac{219}{512}, \frac{1005}{512}, \frac{1121}{512}, 1), A_{14,1}(\frac{31}{64}, \frac{267}{128}, \frac{223}{128}, 1), A_{13,2}(\frac{95}{224}, \frac{439}{224}, \frac{1183}{896}, 1),$$

$$A_{12,3}(\frac{553}{1664}, \frac{17865}{1664}, \frac{2109}{1664}, 1), A_{11,4}(\frac{8495}{23296}, \frac{25911}{23296}, \frac{4295}{3328}, 1), A_{10,5}(\frac{235619}{512512}, \frac{36255}{46592}, \frac{793521}{786432}, 1),$$

$$A_{9,6}(\frac{26231}{64064}, \frac{111}{221}, \frac{21631}{16016}, 1), A_{8,7}(\frac{16435}{54912}, \frac{583}{1408}, \frac{37403}{27456}, 1), A_{7,8}(\frac{4417}{27456}, \frac{4005}{9152}, \frac{1505}{4224}, 1),$$

$$A_{6,9}(\frac{-11815}{128128}, \frac{44273}{128128}, \frac{193275}{128128}, 1), A_{5,10}(\frac{-158381}{512512}, \frac{104805}{512512}, \frac{156503}{73216}, 1), A_{4,11}(\frac{-19}{64}, \frac{4947}{11648}, \frac{34243}{11648}, 1),$$

$$A_{3,12}(\frac{-835}{11648}, \frac{10263}{11648}, \frac{2983}{896}, 1), A_{2,13}(\frac{177}{896}, \frac{1275}{896}, \frac{699}{224}, 1), A_{1,14}(\frac{95}{256}, \frac{471}{256}, \frac{675}{256}, 1),$$

$$A_{0,15}(\frac{219}{512}, \frac{1005}{512}, \frac{1121}{512}, 1).$$

Figure 2 shows the curve on the surface.
7. Automorphisms of the domain

An automorphism of $R^n$ is an invertible linear transformation. After a basis has been chosen, it is given by an $r \times r$ invertible matrix $A$, sending point $P$ to $PA$. In CAGD there are three commonly used interpretations of this.

For a Bézier parametrization of a curve, a point in domain space is $(1 - t, t)$. These are the barycentric coordinates of an affine point relative to the two fixed points 0 and 1. For a Bézier parametrization of a triangular surface patch, a point in domain space is $(1 - s - t, s, t)$. In each case we are using points in $R^r$ whose coordinates sum to 1 as barycentric coordinates in an affine space of dimension $r - 1$. An invertible matrix $A$ that preserves this subset has the property that its row sums are 1.

The other commonly used affine coordinate system uses $r$-tuples $(p_1, \ldots, p_{r-1}, 1)$ as the coordinates of a point, and $(p_1, \ldots, p_{r-1}, 0)$ as the components of a vector. An invertible matrix $A$ that preserves these has the property that its last column is $(0, \ldots, 0, 1)$.

If we interpret $R^r$ as homogeneous coordinates of points of the real projective space $RP^{r-1}$ then $A$ is an arbitrary invertible $r \times r$ matrix.

Whichever setting we are working in, let us now consider the action of $A$ on the domain of another parametrization. Let $f : R^r \rightarrow R^s$ be a homogeneous polynomial transformation of degree $n$. As before let $F$ be the mixed polynomial of $f$ and $A$ be that of $A$. Using these mixed polynomials and the pairing, the composition $f \circ A$ is represented by $\langle A^n/n!, F \rangle$. The coefficients of $f \circ A$, or equivalently the new control points, are given by $\langle X^\alpha/n!, \langle A^n/n!, F \rangle \rangle$. The evaluation of $f(PA)$ is given by $\langle p^n/n!, \langle A^n/n!, F \rangle \rangle$. From the standpoint of forms and sites, this is the end of the story. What follows relates this to
the matrix description of the same computation, and shows the tie-in with representation theory.

The matrix representation formulas for the evaluation of $f(PA)$ for $r = 2$ were used in (Patterson, 1985). The fact that polynomial multiplication is used in the computation of the representation is what suggested the use of the paired algebras.

The linear transformation $A$ of $R^n$ operates on both 1-forms and 1-sites. A 1-form is represented by a column vector and $A$ operates on the left. A point or 1-site is represented by a row vector and $A$ operates on 1-sites on the right.

The automorphism $A$ induces transformations of higher dimensional forms and sites simply by requiring it to preserve the algebra operations of addition, scalar multiplication and multiplication. The actions on the higher dimensional objects are again linear, so are given by matrices.

The matrix $\delta_n(A)$ that describes the action of $A$ on sites of degree $n$ is defined by $P^n\delta_n(A) = (PA)^n$, which as we have seen equals $\langle p, A \rangle^n$.

Similarly if $f$ is a transformation of degree $n$ and associated matrix $B$ as in Section 5, the matrix of $f \circ A$ is $\delta_n(A)B$.

The matrix $\delta_n(A)$ is square of dimension $\binom{n+r-1}{n}$. In fact $\delta_n$ is a representation of $GL(n, R)$ in $GL((n+r-1), R)$.

The following Proposition summarizes the matrix notation for $f(PA)$.

**Proposition 10.** Let $f = (f_1, \ldots, f_s)$ be a parametrization of a Bézier curve, surface or simplex and let $B$ be the associated matrix. Let $A$ be an invertible $r \times r$ matrix. Then

$$f(PA) = P^n\delta_n(A)B$$

We can compute $\delta_n(A)$ using $\langle p, A \rangle^n$.

In the case $r = 2$, $p = p_1 X_1 + p_2 X_2$ and $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. The rows of $A$ written as 1-sites are $r_1 = A_{11} X_1 + A_{12} X_2$ and $r_2 = A_{21} X_1 + A_{22} X_2$. (Since $A$ transforms $R^2$ to itself, we use the symbols $x$ and $X$ for both the domain and the range variables. Confusion is avoided by the convention that in the pairing, the element on the left is always considered to be a site, and that on the right to be a form.) So $\langle p, A \rangle^n = (p_1 r_1 + p_2 r_2)^n$. Thus the rows of $\delta_n(A)$ are found from the binomial expansion.

For $n = 2$,

$$P^2\delta_2(A) = \begin{pmatrix} p_1^2 & 2p_1 p_2 & p_2^2 \end{pmatrix} \begin{pmatrix} A_{11}^2 & 2A_{11} A_{12} & A_{12}^2 \\ A_{11} A_{21} & A_{12} A_{21} + A_{11} A_{22} & A_{12} A_{22} \\ A_{21}^2 & 2A_{21} A_{22} & A_{22}^2 \end{pmatrix}$$
For larger $r$ and $n$, the rows of $A$ determine sites $r_1, \ldots, r_r$ from which we compute the $r^\alpha = r_1^{\alpha_1} \cdots r_r^{\alpha_r}$ for multi-indices $\alpha$ of degree $n$. The $r^\alpha$ are ordered according to the chosen ordering, here lexicographic. The coefficients of $r^\alpha$, again in the chosen ordering, become the rows of $\delta_n(A)$.

Note if $A$ has row sums 1, the same is true of $\delta_n(A)$. If the last column of $A$ is $(0, \ldots, 0, 1)$, the same is true of $\delta_n(A)$.

Figure 3 illustrates the same surface parametrized by $g$ in Figure 2. Superimposed on it is the result of preceeding $g$ by the matrix

$$A = \begin{pmatrix} .8 & .1 & .1 \\ .1 & .8 & .1 \\ .1 & .1 & .8 \end{pmatrix}$$

![Figure 3. Linear transformation of the domain of a surface patch.](image)

This representation of $GL(n)$ is a classical irreducible representation and is discussed in books on the representations of the general linear group. In (Boerner, 1963) it is called the representation on symmetric tensors. The formulas for $\delta_2(A)$ and $\delta_3(A)$ for $r = 2$ are found in a discussion of the representations of $SU(2)$ in (Żelobenko, 1973) and were used in (Patterson, 1985).

8. Conclusion

In this paper we saw how the paired algebras of forms and sites, supplemented by mixed polynomials, provide an elegant way to find the control points for the composition of two polynomial transformations. The case in which the first is a linear transformation was also solved by representation theory, which let us compare the two techniques.
The introduction of sites has at least three three advantages. The first is that it allows the multiplication of points. If you like, this can be thought of as a convenient fiction, much like complex numbers when they were first introduced. Giving a name to the resulting objects of higher degree in the polynomial algebra is not necessary, but the name sites was invented by Ramshaw in order to be brief and concise. Just as the control points of a Bézier simplex can be recovered by pairing with certain fixed sites \( X_\alpha \) in the domain space, so can all of the intermediate points of the de Castlejau algorithm.

The second advantage of sites, seen in the linear case, is that using mixed polynomials instead of matrices saves a great deal of programming bookkeeping to keep track of the order of the basis. By writing the variables instead of relying on the ordering we let the computer algebra system multiply polynomials as usual in whatever order it chooses.

The third advantage of sites is that they allow extension to nonlinear polynomial transformations.

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References


