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1 Abstract

New Riemann-Hilbert method was suggested recently by A. Fokas for studying boundary problems for linear and integrable nonlinear PDEs. In this paper we extend this approach for solution of the vector elastodynamic equation in inhomogeneous geomaterials. Scattering of Rayleigh wave in an elastic quarter space is considered. The suitable Lax pair formulation of the elastodynamic equation is obtained. The integral representations for the solutions of the Lax pair equations are found.

2 Introduction

Two major analytical approaches to wavefield computation in solids are the Fourier and Laplace transform methods. Both of them can be applied to obtain a solution of the classical Lamb’s problem (1) as well as its numerous modifications (e.g., 2). The Laplace transform and the contour deformation act as two major steps of the Cagniard-de Hoop method where the contour has to be deformed to a special one which gives the representation of the solution as the inverse Laplace transform. Even though there is some dissimilarity between the Fourier and Laplace transform methods they share the same analytic idea, i.e., they are associated with separation of variables. That restricts them to boundary value problems of simple geometry, that is a geometry with a single interface.

The need to have at least some exactly solvable models with more than one interface suggests the idea of revisiting the methods based on complex analysis taking into account that they have undergone some essential development in recent years. In particular, the so-called Inverse Scattering Transform (IST), or Riemann-Hilbert method, in the theory of nonlinear integrable PDEs (Soliton Theory) has been greatly developed during the last three decades. Several monographs (e.g., 3,4) can be referred to for comprehensive exposition.

Quite recently an advanced version of the Riemann-Hilbert approach was suggested by A.S. Fokas for solving initial-boundary and boundary value problems for linear and integrable nonlinear PDE’s (5-7). One of the principal methodological ideas of (5-7) is that the basic ingredients of the IST, i.e. the Lax pair and Riemann-Hilbert technique, can be effectively used for solving the initial-boundary and boundary value problems for linear PDE’s. The most significant feature of the approach, which we will call, following (5-7), a unified transform method is that it does not require separation of variables. We believe that this feature may help to solve some of
the long-standing problems which still exist for linear PDEs in many areas of science, and, in particular, in the geophysical applications.

The idea of applying the unified transform method to solve seismological problems seems attractive not only because of the possibility of solving diffraction and scattering problems of nonstandard geometry, but also because it may be applied to various nonlinear versions of the elastodynamic equation. The transition from linear to nonlinear integrable PDE goes through the Lax pairs, so it seems important to introduce the Lax pair approach into mathematical geophysics.

In this paper Lax pair formulation for the vector elastodynamic equation was developed. The matrix Lax Pair equation was diagonalized and two independent scalar Lax Pair systems of equations were obtained. Solution to those systems was constructed in two separate complex domains. The boundary conditions for a quarter-plane were incorporated into the solution. The results of joint uniformization of these solutions by the Jacobian elliptic functions, which is required in order to write the final integral representaion, will be reported at the meeting.

3 A Unified Transform (Riemann-Hilbert) Method

In this section, the Riemann-Hilbert method suggested in (5-7) for solving initial-boundary and boundary value problems for linear PDE’s (and integrable nonlinear PDE’s) is briefly described.

Let \( q(x, y) \) satisfy the linear PDE with constant coefficients,

\[
L(\partial_x, \partial_y)q = 0,
\]

where \( L(\partial_x, \partial_y) \) is a linear operator of \( \partial_x \) and \( \partial_y \) (of the first order with respect to either \( \partial_x \) or \( \partial_y \)) with constant coefficients. The general features of the scheme of solving an initial-boundary (parabolic or hyperbolic equation) or a boundary (elliptic) problem for (3.1) are:

**Lax pair formulation.** As it is shown in (5-7), equation (3.1) can be written as a compatibility condition, i.e. \( \mu_{xy} = \mu_{yx} \) of the two linear ODEs,

\[
\mu_x + ik\mu = Q(k, q, q_x, q_y, ...),
\]

\[
\mu_y + \omega(k)\mu = \bar{Q}(k, q, q_x, q_y, ...)
\]

for an auxiliary scalar function \( \mu(x, y, k) \). In (3.3), \( \omega(k) \) is a root of the dispersion equation,

\[
L(-ik, -\omega(k)) = 0.
\]

(3.2) and (3.3) form a Lax pair for the original PDE (3.1).

**Riemann-Hilbert problem.** A straightforward analysis of the general solution of system (3.2) and (3.3) allows one to single out its particular solution, \( \mu(x, t, k) \), which is an analytic function on \( k \in C\setminus K \), where \( K \) is a certain directed contour on the complex \( k \)-plane. The boundary values, \( \mu_{\pm}(x, t, k) \), of the function \( \mu(x, t, k) \) on the contour \( K \) satisfy the jump conditions,

\[
\mu_+(x, t, k) - \mu_-(x, t, k) = e^{-ikx - \omega(k)y} \rho(k),
\]

(3.5)
which constitute a certain Riemann-Hilbert problem (RH). The contour $K$ and the jump function $\hat{\rho}(k)$ are uniquely specified by the type of concrete initial-boundary or boundary value problem posed for the original PDE (3.1) and by the relevant initial-boundary data.

Having solved the RH problem (3.5), one can reconstruct the solution $q(x, y)$ of (3.1) using this time the system (3.2), (3.3) evaluated at some special points $k$ (usually, at $k = \infty$) as an algebraic equation for $q$.

**Solution for the Riemann-Hilbert problem.** The Riemann-Hilbert problem (3.5) admits an explicit solution via the Cauchy integral:

$$\mu = \frac{1}{2\pi i} \int_{K} \frac{e^{-ik'x - \omega(k')y} \hat{\rho}(k')}{k' - k} dk',$$

which yields the integral representation,

$$q(x, y) = \int_{K} e^{-ik'x - \omega(k')y} \rho(k) dk,$$

for the function $q(x, t)$. The function $\rho(k)$ is given by an elementary combination of $\hat{\rho}(k)$ and the parameter $k$ which is determined by the rhs’s of the Lax pair.

The most important advantage of the RH method is that it provides a unified algorithmic approach to determining the contour $K$ and function $\rho(k)$ in representation (3.7) through the given initial-boundary data without any prior restrictions on the domains, the type of operator $L$, or the type of boundary conditions.

## 4 Scattering of Rayleigh wave in an elastic quarter space

Rayleigh wave scattering in a corner is of special interest to seismologists and engineers as a preliminary step to understanding propagation through interfaces of different materials and as a base to understanding scattering at step-like inhomogeneities of stress-free surfaces. The problem has been studied by many authors during the last decades (e.g., 8 - 10). We are using the quarter-space problem as a model problem for developing the method.

The problem is two-dimensional in $xz$ plane, and $u$ and $w$ are the $x$ and $z$ components of displacement. The elastodynamic equation can be written in the form of (3.1) as follows

$$Lq = 0, \quad L = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & -1 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 & -1 \\ h^2 l^2 + h^2 \partial_{zz} & 0 & (l^2 - h^2) \partial_x & h^2 \partial_x \\ 0 & h^2 l^2 + l^2 \partial_{zz} & (l^2 - h^2) \partial_x & h^2 \partial_x \end{pmatrix},$$

where $q$ is a vector with the coordinates $q_1 = u$, $q_2 = w$, $q_3 = \partial_x u$, $q_4 = \partial_x w$, $h^2 = \frac{\rho \omega^2}{\lambda + 2\mu}$, $l^2 = \frac{\rho \omega^2}{\mu}$, and $\rho$, $\lambda$, $\mu$ and $\omega$ are density, Lamé constants and frequency respectively. On the surfaces $z = 0$ and $x = 0$ the stress conditions are:

$$X_z = \mu \left( \frac{\partial}{\partial z} u + \frac{\partial}{\partial x} w \right) = -X_z^{inc}, \quad Z_z = \lambda \frac{\partial}{\partial x} u + (\lambda + 2\mu) \frac{\partial}{\partial z} w = -Z_z^{inc}, \quad z = 0 \quad (4.9)$$

$$X_x = \mu \left( \frac{\partial}{\partial z} w + \frac{\partial}{\partial x} u \right) = -X_x^{inc}, \quad Z_x = \lambda \frac{\partial}{\partial x} w + (\lambda + 2\mu) \frac{\partial}{\partial z} u = -Z_x^{inc}, \quad x = 0 \quad (4.10)$$

where $X_z^{inc}$, $X_x^{inc}$, $Z_z^{inc}$, $Z_x^{inc}$ denote the given stresses of the incident Rayleigh wave.
4.1 Lax Pair

Employing the approach developed in (7) we obtained the matrix Lax pair system of equations corresponding to (4.8) in the following form

\[(\partial_z - ik)\nu = q\]
\[L\nu = 0\]

where \(k\) is a spectral parameter, \(\nu\) is a vector analogy of the auxiliary function \(\mu\) (3.2) Using the first equation (4.11) to exclude differentiation with respect to \(z\) from the second one we obtain

\[(\partial_z - ik)\nu = q\]
\[\partial_z \nu + i\Omega \nu = \bar{L}q,\]

where

\[i\Omega = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ h^2(k^2 - h^2) & 0 & 0 & -ik(k^2 - h^2) \\ 0 & l^2(k^2 - h^2) & h^2 & 0 \end{pmatrix} \]

\[\bar{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -h^2(k^2 + ik) & 0 & 0 & (l^2 - h^2) \\ 0 & -l^2(k^2 + ik) & h^2 & 0 \end{pmatrix} \]

Diagonalization of \(\Omega\) leads to following Lax pair system

\[\partial_z M - ikM = T^{-1}q\]
\[\partial_z M + \Lambda M = T^{-1}\bar{L}q\]

where \(M = T^{-1}\nu\) and \(\Lambda\) is a diagonal matrix of the eigen values of \(i\Omega\) such that

\[\Lambda_1 = \sqrt{k^2 - h^2}, \ \Lambda_2 = -\Lambda_1, \ \Lambda_3 = \sqrt{k^2 - l^2}, \ \Lambda_4 = -\Lambda_3\]

The inverse matrix \(T^{-1}\) has the following form

\[T^{-1} = \frac{1}{2} \begin{pmatrix} \frac{l^2 - k^2}{l^2} & \frac{ik\sqrt{k^2-h^2}}{h^2} & \frac{\sqrt{k^2-h^2}}{h^2} & -\frac{ik}{l^2} \\ \frac{l^2 - k^2}{l^2} & -\frac{ik\sqrt{k^2-h^2}}{h^2} & -\frac{\sqrt{k^2-h^2}}{h^2} & \frac{ik}{l^2} \\ \frac{k^2}{l^2} & \frac{ik(k^2-h^2)}{h^2\sqrt{k^2-l^2}} & \frac{k^2}{h^2\sqrt{k^2-l^2}} & \frac{ik}{l^2} \\ \frac{k^2}{l^2} & \frac{ik(k^2-h^2)}{h^2\sqrt{k^2-l^2}} & \frac{k^2}{h^2\sqrt{k^2-l^2}} & \frac{ik}{l^2} \end{pmatrix} \]

The matrix \(T^{-1}\) (4.17) can be represented as follows

\[T^{-1} = k^2X_2 + kX_1 + X_0 + O\left(\frac{1}{k}\right), \text{ as } k \to \infty, \ \Lambda_1 = k + ..., \ \Lambda_3 = k + ...\]
where the matrices $X_{0-2}$ depend only on the elastic parameters. Solving the first equation of the Lax pair (4.15) as a linear differential equation with respect to $z$, and then integrating the solution twice by parts as well as using (4.18) we can obtain the following asymptotic formula for $M$

$$ \begin{align*}
M &= ikX_{2q} + iX_{1q} + (X_{2q})_z + \frac{iX_0 + (X_{1q})_z - i(X_{2q})_{zz}}{k} + O\left(\frac{1}{k^2}\right) 
\end{align*} \tag{4.19} $$

Therefore, in order to have a solution which decays as $k \to \infty$ we choose a new vector function $N$ as

$$ \begin{align*}
N &= M - ikX_{2q} - iX_{1q} - (X_{2q})_z. 
\end{align*} \tag{4.20} $$

![Diagram of S1, S2, S3](image)

**Figure 1:** $S_1$, $S_2$, $S_3$ are the areas of boundness of the solutions $\phi_1$, $\phi_2$, $\phi_3$ respectively.

In terms of this function the Lax pair (4.15) takes the form

$$ \begin{align*}
N_x - ikN &= (X - k^2X_2 - kX_1)q - i(X_{1q})_z - (X_{2q})_{zz} \tag{4.21} 
\end{align*} $$

$$ \begin{align*}
N_x + \Lambda N &= (X\Lambda - ik\Lambda X_2 - i\Lambda X_1)q - ik(X_{2q})_z - i(X_{1q})_z - (X_{2q})_{xx} - \Lambda(X_{2q})_z 
\end{align*} $$

where $X = T^{-1}$. 

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4.2 Uniformization of the scalar Lax Pairs

One can see that only the first and the third scalar pairs of (4.21) corresponding to $\Lambda_1$ and $\Lambda_3$ are truly independent:

\[
\phi_z - ik \phi = Q_1 \quad \psi_z - ik \psi = Q_3 \\
\phi_x + \Lambda_1 \phi = Q_1 \quad \psi_x + \Lambda_3 \psi = Q_3
\]  
(4.22)

where $\phi = N_1$ and $\psi = N_3$, and $Q_1, \tilde{Q}_1, Q_3$ and $\tilde{Q}_3$ are the corresponding coordinates of the right-hand side parts of (4.21). In particular

\[
Q_1 = \frac{1}{2} q_1 + \frac{ik}{h^2} (-k + \sqrt{k^2 - h^2})q_2 + \frac{1}{l^2} (-k + \sqrt{k^2 - h^2})q_3 - \frac{i}{h^2} \bar{q}_{13z} - \frac{1}{l^2} q_{1zz} - \frac{i}{h^2} \bar{q}_{2zz}
\]  
(4.23)

We already took care about the decay of $\phi$ as $k \to \infty$ and $\Lambda_1 = k + \ldots$. Here and below we will use the notation $k \to \infty^+$ for this case. However, it follows from (4.22,4.23) that as $k \to \infty^-$ (that is $\Lambda_1 = -k + \ldots$) then

\[
\psi = a_1 k + a_0 + a_{-1} k^{-1} + \ldots \quad \text{where} \quad a_1 = \frac{q_2}{h^2}
\]

To avoid this growth of the auxiliary function one has to introduce a new $\tilde{\phi}$ as

\[
\tilde{\phi} = \phi + \frac{1}{2h^2} (\sqrt{k^2 - h^2} - k)q_2
\]

Finally, dropping the tilde, going back from $q_{1-4}$ to the displacements $u$ and $w$ and introducing $\tau_1 = \frac{1}{2} (u_x + w_x)$ we obtain the first scalar Lax Pair in the following form

\[
\phi_z - ik \phi = \frac{1}{h^2} (\sqrt{k^2 - h^2} - k)\tau_1 - \frac{1}{h^2} (\tau_{1x} + i\tau_{1z})
\]  
(4.24)

\[
\phi_x + \sqrt{k^2 - h^2} \phi = \frac{i}{h^2} (k - \sqrt{k^2 - h^2})\tau_1 - \frac{i}{h^2} (\tau_{1x} + i\tau_{1z})
\]

In a similar way the third Lax Pair takes the form

\[
\psi_z - ik \psi = \frac{1}{l^2} (\sqrt{k^2 - l^2} - k)\tau_2 - \frac{1}{l^2} (\tau_{2x} + i\tau_{2z})
\]  
(4.25)

\[
\psi_x + \sqrt{k^2 - l^2} \psi = \frac{i}{l^2} (k - \sqrt{k^2 - l^2})\tau_2 - \frac{i}{l^2} (\tau_{2x} + i\tau_{2z})
\]

where $\tau_2 = \frac{1}{2} (u_x - u_z)$.

The auxiliary functions $\phi$ and $\psi$ satisfy the following asymptotic representations

\[
\phi = O\left(\frac{1}{k}\right) \quad k \to \infty^+
\]  
(4.26)

\[
\phi = -2i \frac{1}{h^2} \tau_1 + O\left(\frac{1}{k}\right) \quad k \to \infty^-
\]

and

7
\[ \psi = O\left(\frac{1}{k}\right) \quad k \to \infty^+ \tag{4.27} \]
\[ \psi = -\frac{2i}{\ell^2} \tau_2 + O\left(\frac{1}{k}\right) \quad k \to \infty^- \]

Introducing for the first scalar Lax Pair a new spectral parameter \( \xi \) as follows
\[ k = \frac{\hbar}{2}(\xi + \frac{1}{\xi}) \tag{4.28} \]

and following the method developed in (7) for the Helmholtz equation we obtain three solutions \( \phi_1, \phi_2, \phi_3 \) of the first scalar Lax Pair equations in terms of the integrals of \( Q_1, Q_1 \). The functions \( \phi_1, \phi_2, \phi_3 \) are bounded in the regions \( S_1, S_2, S_3 \) respectively of the complex \( \xi \) plane as shown in Fig.1.

![Contour of integration and the jump functions \( \rho_{ij} \) of the first scalar Lax pair](image)

Figure 2: Contour of integration and the jump functions \( \rho_{ij} \) of the first scalar Lax pair

The contour of integration \( K \) and the jump functions \( \rho_{ij} \) are indicated in Fig.2. The similar result can be obtained for the third scalar Lax Pair using the following uniformization
\[ k = \frac{l}{2}(\zeta + \frac{1}{\zeta}) \tag{4.29} \]
The jump functions $\rho_{ij}$ are defined by the boundary values of the functions $u(x, z)$ and $w(x, z)$. Not all of these values, however, can be determined via the boundary conditions (4.9, 4.10). To determine the unknown parts of $\rho_{ij}$, one can use the global relation (cf. (5 - 7)),

$$\rho_{13} + \rho_{23} = 0.$$  

Indeed, from this relation, written for each of the Lax pairs, we obtain an underdetermined system of two algebraic equations,

$$\sum_{j=1}^{4} C_j^{(1)}(\xi)\Psi_j^{(1)}(\xi) = F^{(1)}(\xi) - \Theta^{(1)}(\xi),$$

and

$$\sum_{j=1}^{4} C_j^{(2)}(\zeta)\Psi_j^{(2)}(\zeta) = F^{(2)}(\zeta) - \Theta^{(2)}(\zeta),$$

where $C_j^{(k)}$ are certain rational functions and $F^{(k)}, \Theta^{(k)}$ are given explicitly in terms of the known boundary condition terms $X^{inc}, Z^{inc}$ (4.9, 4.10). The functions $\Psi_j^{(k)}$ are unknown. Taking into account several transformations of the complex $\xi$ and $\zeta$ planes which leave the corresponding Lax pairs invariant and using the joint uniformization of the square roots $\Lambda_1$ and $\Lambda_3$ by the Jacobian elliptic functions, one can reduce the number of unknown functions $\Psi_j^{(k)}$ and write the final integral representation (3.7). The results of this part will be reported at the next paper.

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6 Bibliography


