2016 IUPUI HIGH SCHOOL MATH CONTEST SOLUTIONS

1. PROBLEM 1

Martian tic-tac-toe is played on a 4×4 *grid. The goal is to make a four in a row, but there is an extra move available: Instead of adding a symbol to the grid, you can chose to exchange all Xs for Os and all Os for Xs. This move is available only if your opponent did not swap immediately before you, and it is not available for the first move of the game. Show that the second player player has a strategy to never lose (remember there can be ties).*

The exact mechanism of 'four-in-a-row' play is not relevant; let us list the properties of the game that are truly necessary for a solution:

- Two players alternate turns.
- On her turn, a player changes the configuration of the game. (Exchanging Xs and Os IS a configuration change!).
- Some moves are automatic wins (in this concrete game, reaching any four-in-a-row).
- Exchanging Xs and Os reverses the player that wins on a given winning configuration (if a configuration was winning for player A, now it is winning for player B, and the other way around).
- There is no wining move from the initial configuration (otherwise player B would lose before having a chance to enact a strategy). Moreover, exchanging Xs and Os is not a valid move from the initial configuration.

Under these conditions, what can player B do? Either she has a move that creates a four-in-a-row, in which case she should make that move and win; or else she does not have an immediately winning move, in which case she should exchange Xs and Os. This leaves player A with a configuration that has no immediately winning move, so no matter what A plays, B gets one more turn. In other words, B cannot lose.

Notice that this strategy is not available to player A because she cannot start with a symbol exchange. After A makes a first move, the strategy above has B exchanging every single turn until she can make a winning move. In effect, B takes control of the exchange move so A can never make that move, and consequently it is always available for B's next turn.

2. Problem 2

Consider a convex polygon with 2016 *sides (sides can have different lengths, but angles are all under 180 degrees). What is the largest possible number of obtuse angles? What is the largest possible number of acute angles?*

Recall that the sum of angles of a convex polygon with n sides is equal to $(n-2)180°$. Indeed, pick a point, say X, inside the polygon and connect it to each vertex. This way you will get n triangles. The sum of their vertices is $n180°$. Therefore, the sum we are looking for is equal to

$$n180° - \text{the sum of angles adjacent to } X = n180° - 360° = (n-2)180°.$$

Thus, each vertex of the regular polygon with 216 sides has an angle $(214/216)180°$, which is obtuse. That is, the largest possible number of obtuse angles is 216.

To compute the largest number of acute angles, call the polygon P. Lets define the "external angle" of P at a vertex V as follows: Looking at V from within the polygon, extend the left edge so it forms with the right edge a complementary angle on the outside of P. Notice that the internal and external angles at V add up to $180°$. Therefore, the sum of all external angles in a convex polygon is ALWAYS

$$n180° - \text{sum of internal angles} = n180° - (n-2)180° = 360°.$$

If four or more internal angles were acute, the corresponding external angles would all be larger than $90°$ each, so the total sum would exceed $360°$. This is impossible, so at most 3 internal angles can be acute. This number CAN be achieved in the following manner: Start with a quadrilateral that has 3 acute angles (for example take a square and push one vertex inward along its diagonal). The fourth angle, which is obtuse, can be "shaved", leaving two new obtuse angles. This process can be repeated on any obtuse angle indefinitely, so we can reach a total of 2013 obtuse angles, plus 3 acute ones that remain unaltered throughout.

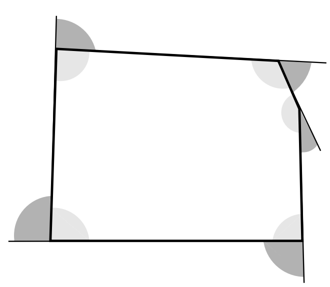

FIGURE 1. Exterior (darker shaded) and interior (lighter shaded) angles of a 5-gon with 3 acute angles.

3. Problem 3

For the 4th of July, a bowling alley will paint each of its bowling pins in either red, white, or blue. The pins are set up in the usual triangular configuration. How many different arrangements include all three colors, assuming that reflections and rotations of a given arrangement are considered the same?

Ignoring the "all colors must be used" restriction and "arrangements must be distinct under rotations and reflections" restriction, there are

$$n^{10}$$

ways to color 10 pins with n colors. Now, if the arrangement is not by itself symmetric with respect to some rotation or some reflection, it will have five more equivalent ones, that is, it appears 6 times as a non-distinct arrangement. For each symmetry the arrangement satisfies this count reduces by 1. As there are

$$3n^6$$

arrangements symmetric with respect to reflections (3 reflections and for each you need pick the colors only for 6 pins as other 4 will be determined by symmetry), and there are

$$2n^4$$

arrangements symmetric with respect to rotations (2 rotations and for each you need pick the colors only for 4 pins as other 6 will be determined by symmetry), the total number of different arrangements using at most n colors is

$$(n^{10} + 3n^6 + 2n^4)/6 = c(n).$$

Now, to get the answer to the problem, we need to remove from $c(3)$ colorings that use only 1 or 2 colors. Configurations that use only one color:

$$3c(1)$$

as you have 3 option from which you pick a color. Configurations that use at most two colors:

$$3c(2)$$

as you have 3 pairs of colors to choose. Configurations that use exactly two colors:

$$3\big(c(2) - 2c(1)\big)$$

as for each choice of a pair of colors you need to remove all the configurations with only the first and only second color picked. Finally, configurations that use all three colors:

$$c(3) - 3(c(2) - 2c(1)) - 3c(1) = 9612.$$

4. PROBLEM 4

Martian pool is played on a regular pentagonal table with one ball and no pockets. The ball always bounces off edges according to the equal angles rule, and is so small that it can be represented by a point. A "good shot" involves bouncing off two consecutive edges, but depending on the initial ball position, some good shots may not be possible.

a) *Show that some good shot is possible from any initial ball position.*
b) *Find the region R of initial positions from where a good shot is possible on any pair of consecutive edges.*
c) *Find an initial ball position within the region R, and a direction that results in a "very good shot"; i.e., a shot in which the ball bounces off all five edges in consecutive order.*

We consider the external angle of the pentagon $\mathscr{P}$; see Figure 2: $\alpha = \left(\frac{360}{5}\right)^\circ = 72^\circ$, and consequently, $\beta = 180^\circ - \alpha = 108^\circ$. Moreover, since the diagonal is parallel to the left edge of $\mathscr{P}$ (by symmetry), the angle marked $*$ also equals α, and γ becomes $108^\circ - \alpha = 36^\circ$.

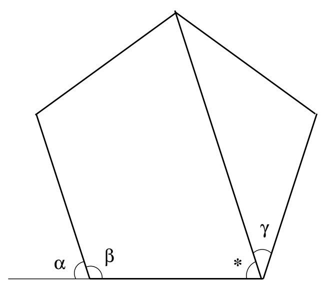

FIGURE 2.

Take a look at Figure 3; the edge B and the diagonal form equal angles (both α) with edge A, so any shot bouncing on A from within the shaded triangle will miss B. Conversely, a shot from outside the shaded triangle with a low enough angle, will bounce off A onto B.

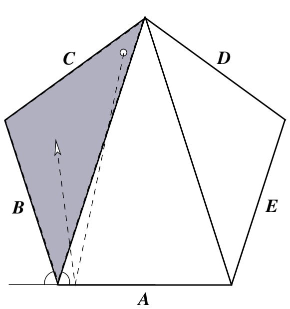

FIGURE 3.

A similar situation holds on the right side of the table, so there is a good shot bouncing off A (toward either B or E) from any initial position, solving part a). In fact, a good shot bouncing off A in either direction is possible as long as it starts in the isoceles triangle with base A. The same is true for every edge, so a good shot involving any pair of consecutive edges is possible from within the intersection of all five such triangles. This is the small pentagon shaded in Figure 4.

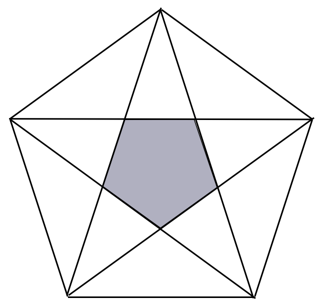

FIGURE 4.

Now consider Figure 5. The equal angles bounce rule means that a straight line that continues into the reflection of $\mathscr{P}$ will reflect back into a valid trajectory. This is equally valid after multiple reflections, so we can unfold multiple reflected copies of $\mathscr{P}$ (keeping track of the order reversal among edges), and find a straight line from the small pentagon T to the 5-fold reflected edge that completes a very good shot.

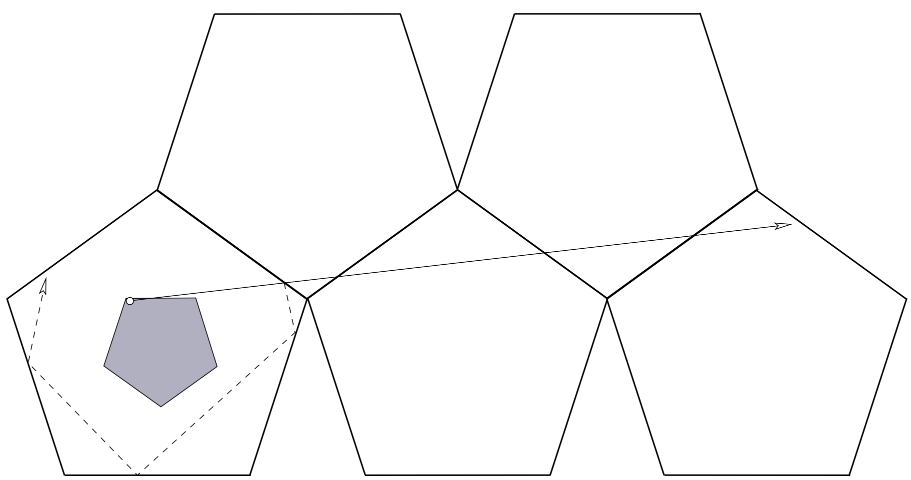

FIGURE 5.

5. Team Problem

The point with coordinates $(14,20)$ *is "hidden" because it cannot be seen from the origin* $(0,0)$ *(the point* $(7,10)$ *blocks the view). The square formed by* $(14,20)$*,* $(14,21)$*,* $(15,20)$*, and* $(15,21)$ *is "hidden" because all its four points are hidden.*

a) *Find the* 3×3 *hidden square (all its* 9 *points are hidden) that is closest to the origin.*
b) *Now turn to* 3*-dimensional integer coordinates* (a,b,c)*, and find the* $2\times 2\times 2$ *hidden cube (all its* 8 *points are hidden) that is closest to the origin.*

The Chinese remainder theorem is very handy for finding a solution to this problem (see, https://en.wikipedia.org/wiki/Chinese_remainder_theorem). For the 3×3 case, you can find a solution by finding (m,n) such that

$$\begin{cases} m = 0\ (mod\ 2), & m = 0\ (mod\ 3), & m = 0\ (mod\ 2); \\ m+1 = 0\ (mod\ 5), & m+1 = 0\ (mod\ 7), & m+1 = 0\ (mod\ 11); \\ m+2 = 0\ (mod\ 2), & m+2 = 0\ (mod\ 13), & m+2 = 0\ (mod\ 2); \end{cases}$$

and

$$\begin{cases} n = 0\ (mod\ 2), & n = 0\ (mod\ 5), & n = 0\ (mod\ 2); \\ n+1 = 0\ (mod\ 3), & n+1 = 0\ (mod\ 7), & n+1 = 0\ (mod\ 13); \\ n+2 = 0\ (mod\ 2), & n+2 = 0\ (mod\ 11), & n+2 = 0\ (mod\ 2); \end{cases}$$

If such a pair is found (and, by the CRT, m and n both exist less than $2\times 3\times 5\times 7\times 11\times 13 = 30030$), then

$$(m,n),(m+2,n),(m,n+2),\ \text{ and }\ (m+2,n+2)$$

are all invisible because each coordinate is even;

$$(m,n+1)$$

is invisible because each coordinate is a multiple of 3;

$$(m+1,n)$$

is invisible because each coordinate is a multiple of 5;

$$(m+1,n+1)$$

is invisible because each coordinate is a multiple of 7;

$$(m+1,n+2)$$

is invisible because each coordinate is a multiple of 11;

$$(m+2,n+1)$$

is invisible because each coordinate is a multiple of 13. The smallest case that satisfies these conditions is

$$m = 28,104 \quad \text{and} \quad n = 4,640.$$

At this point, it is likely that the best approach is to come up with a for-next-loop approach and use a computer to check possibilities. The correct answer is (1274, 1308) and uses the primes 2, 3, 5, 7, 11, and 17.

For the $2\times2\times2$ case, however, the answer is much larger, and finding it requires many more improvements in efficiency.

First, its useful to have an upper bound in mind, so finding any (presumably suboptimal) solution is a good start. Using the Chinese Remainder Theorem, there are $70 = 8C4$ integers between 0 and $9,699,689 (= 2\times3\times5\times7\times11\times13\times17\times191)$ which are congruent to zero for exactly four of the first 8 primes and are congruent to -1 for the other four. This leads quickly to a heuristic solution

$$(a,b,c) = (573495, 728364, 766479).$$

This has a radius of approximately 1.2 million. Using other sets of 8 low primes will improve this somewhat, but there will still be a lot of cases to check on the order of $(10^6)^3$, or 10^{18}. At a rate of a billion cases checked per second, this would be solved in approximately 31.7 years. Noting that order doesnt matter divides the required calculation by 6 to just over 5 years. Clearly, additional efficiency is required.

This is not hard to do, however, because you only need to look at numbers which are multiples of at least four different primes. Actually, to be even more precise, you only need to look at adjacent PAIRS of numbers which are multiples of at least four different primes. These start out very sparse! The smallest such pair is 7314 and 7315. The second is 8294 and 8295. The third is 8645 and 8646. The fourth is 9009 and 9010, and these are the only examples below $10,000$. There are a total of 780 such pairs less than $100,000$, and a total of $27,659$ below 1 million. While still requiring a computer, this is much more tractable. The optimal values turn out to be:

$$107,184/107,185; \quad 196,910/196,911; \quad \text{and} \quad 329,120/329,121.$$

The 8 primes involved are 2 (obviously), $3,5,11,13,17,29$, and 97.