(1) Let $M$ be a smooth, compact $n$–dimensional manifold without boundary ($n > 0$) and let $N$ be a smooth manifold (also without boundary). Assume that there exists a submersion $p: N \to \mathbb{R}$.

Prove that for each smooth function $f: M \to N$, there exist at least two distinct points $x, y \in M$ such that $T_xf: T_xM \to T_{f(x)}N$ and $T_yf: T_yM \to T_{f(y)}N$ are not surjective.

(2) Let $M$ be a smooth manifold without boundary. Let $X$ be a $C^\infty$ vector field on $M$, and let $f, g: M \to \mathbb{R}$ be $C^\infty$ functions.

a) State the definition of the Lie derivative $\mathcal{L}_X(g)$.

b) Prove, directly from your definition in part a), that if $\mathcal{L}_X(g) = 0$ then $\mathcal{L}_{fX}(g) = 0$ as well.

(3) (a) Prove that $H = \{(v, w) \in \mathbb{R}^n \times \mathbb{R}^n : \langle v, w \rangle = 1\}$ is a smooth manifold. Here $\langle v, w \rangle = v \cdot w$ is the standard inner product on $\mathbb{R}^n$.

(b) Is $H$ transverse to the diagonal $\Delta = \{(v, v) \in \mathbb{R}^n \times \mathbb{R}^n\}$? Prove your answer.

(4) Consider the vector field $\frac{\partial}{\partial x_1}$ on $\mathbb{R}^2$. Let $\psi: S^2 \setminus \{N\} \to \mathbb{R}^2$ be the stereographic projection map

$$\psi(x, y, z) = \left(\frac{x}{1 - z}, \frac{y}{1 - z}\right)$$

(where $N = (0, 0, 1)$).

(a) Show that the vector field $V = \psi^\ast(\frac{\partial}{\partial x_1})$ extends to a smooth vector field $X$ on the entire sphere $S^2$, with the property that $X_p = 0$ if and only if $p = N$.

(b) Let $\gamma: (-a, a) \to S^2$ be an integral curve of $V$. Show that $\gamma$ extends to an integral curve defined on all of $\mathbb{R}$, and prove that

$$\lim_{t \to \infty} \gamma(t) = \lim_{t \to -\infty} \gamma(t) = N.$$

(5) Let $X$, $Y$, and $Z$ be compact oriented $k$–dimensional manifolds without boundary, and consider smooth maps $f: X \to Y$ and $g: Y \to Z$.

Prove that

$$\deg(g \circ f) = \deg(g) \deg(f).$$

(6) Let $A$ be a $2 \times 2$ matrix with real entries, and consider the 1–form $\omega_A$ on $S^1$ defined by $\omega_A(v) = \langle v, Ax \rangle$, where $v \in T_x(S^1) \subset \mathbb{R}^2$.

(a) Give a formula for $\int_{S^1} \omega$ in terms of the entries of $A$.

(b) Characterize those matrices $A$ for which the form $\omega_A$ is closed but not exact.