IUPUI Qualifying Exam in Real Analysis, Winter 2013

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Problem 1. Let A, B and C be measurable sets on the line such that

$$m(A \cap B) = 0.9, \quad m(A \cap C) = 0.9, \quad m(B \cap C) = 0.9, \quad m(A \cup B \cup C) = 1.$$

Prove that

$$m(A \cap B \cap C) \geq 0.85.$$

Problem 2. Suppose that a sequence $E_1, E_2, \dots$ of measurable sets on $[0,1]$ satisfies the Cauchy condition, so that for every $\varepsilon > 0$ there exists $N > 0$ such that

$$m(E_n \,\Delta\, E_m) \leq \varepsilon,$$

for all $n, m \geq N$, where $A \,\Delta\, B = (A \setminus B) \cup (B \setminus A)$. Prove that there exists a measurable set E on $[0,1]$ such that

$$\lim_{n\to\infty} m(E_n \,\Delta\, E) = 0.$$

Problem 3. Evaluate the limit,

$$\lim_{n\to\infty} \int_0^1 \frac{nx^n dx}{(1+x^n)^2},$$

and justify your answer.

Problem 4. Let us enumerate all rational points on $[0,1]$,

$$\mathbb{Q} \cap [0,1] = \{r_1, r_2, \dots\}.$$

Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{|x - r_n|}{n^{3/2}}$$

on [0,1]. Prove that the function $f(x)$ is differentiable almost everywhere.

Problem 5. Prove that if f_n converges to f in $L^p(E)$ and g_n converges to g in $L^q(E)$, where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \qquad p > 0, \quad q > 0, \quad r \geq 1,$$

then $f_n g_n$ converges to fg in $L^r(E)$.

Problem 6. Let

$$f_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k^{2/3}}.$$

Prove that there exists an integrable function $f(x)$ on the interval $[0, 2\pi]$ such that

$$\lim_{n\to\infty} \int_{[0,2\pi]} |f(x) - f_n(x)| dx = 0.$$