Problem 1. (15)
(a) Show that in a domain principal prime ideals are maximal among principal ideals.
(b) Show that if $d \in \mathbb{Z}$ is not a square of an integer, then any non-zero prime ideal (not necessarily principal) in the quadratic domain $\mathbb{Z}[\sqrt{d}]$ is maximal.
Problem 2. (10)
Let $M$ be a maximal ideal in a commutative ring $R$ with identity and $n$ be a positive integer. Show that $R/M^n$ has a unique prime ideal, where $M^n$ is the ideal of $R$ generated by the set 
\[
\{x_1x_2\cdots x_n | x_1, x_2, \ldots, x_n \in M\}.
\]
Problem 3. (10)

Factor $19 + 4i$ as a product of irreducibles in the ring $\mathbb{Z}[i]$. Justify your answer.
Problem 4. (10)
Let $f$ be an irreducible polynomial of degree $n$ over the field $F$. Let $g$ be any polynomial of positive degree in $F[x]$. Show that any irreducible factor of $f \circ g$ has degree divisible by $n$. 
Problem 5. (15)
Let $\zeta$ be a primitive 7th root of unity in $\mathbb{C}$. Determine the following with proofs.
(a) The galois group $\text{Aut}_{\mathbb{Q}} \mathbb{Q}(\zeta)$;
(b) All intermediate subfields of the extension $\mathbb{Q}(\zeta)/\mathbb{Q}$. 
Problem 6. (15)
Show that any group $G$ with $|G| = 105$ is solvable.
**Problem 7. (20)** Let \( G \) be a finite abelian group (written multiplicatively) and let \( p \) be a prime. Define the subgroups: 
\[ G^p = \{ a^p | a \in G \} \] and 
\[ G_p = \{ x \in G | x^p = 1 \} \].

(a) Prove that \( G/G_p \cong G^p \) and \( G/G^p \cong G_p \).

(b) Prove that the number of subgroups of \( G \) of order \( p \) equals the number of subgroups of \( G \) of index \( p \).