Complex Analysis Qualifying Exam
IUPUI Winter 2016
(exam by R. Pérez)

1. (10 points) Prove that the function $f(z) = |z^2|$ has a complex derivative at $z = 0$, and at no other point.

2. (10 points) Compute the improper integral below, justifying your methodology:

$$\int_{-\infty}^{\infty} \frac{x^2\,\mathrm{d}x}{x^4 + 5x^2 + 4}.$$

3. (10 points) Show that if $|b| < 1$, then the polynomial $z^3 + 3z^2 + bz + b^2$ has exactly two roots (counted with multiplicity) in the unit disk.

4. (10 points) Find a conformal map that takes the region $\big\{|z| < 1\,,\ |z-1/2| > 1/2\big\}$ in a one-to-one fashion onto the unit disk.

5. (10 points) Prove that if f is entire and satisfies $|f(z^2)| \le |f(z)|$ for all $z \in \mathbb{C}$, then f is constant.

6. (10 points) Show that the function $\frac{1}{2}\big(z + 1/z\big)$ maps the half-disk $\big\{|z| < 1\,,\ \operatorname{Im} z < 0\big\}$ in a one-to-one fashion onto the upper half-plane $\big\{\operatorname{Im} z > 0\big\}$.

7. (10 points) Let $\mathbb{D}$ denote the unit disk, and suppose the holomorphic function $g : \mathbb{D} \longrightarrow \mathbb{D}$ has two distinct fixed points. Prove that $g(z) \equiv z$.

QUALIFYING EXAM
COMPLEX ANALYSIS (MATH 53000)
JANUARY 2016

In what follows, $\mathbb{D} := \{z : |z| < 1\}$ and $\mathbb{T} := \{z : |z| = 1\}$.

1. Given a circle $C = \{z : |z - a| = R\}$, find two points symmetric with respect to both C and the imaginary axis.

2. Let f be a holomorphic function in $\mathbb{D}$ which is continuous in $\overline{\mathbb{D}}$. Assuming that $|f(z)| > |f(0)|$ for all $z \in \mathbb{T}$, show that f has at least one zero in $\mathbb{D}$.

3. If P is a polynomial of degree n and $|P(z)| \leqslant M$ for $z \in \mathbb{D}$, then $|P(z)| \leqslant M|z|^n$ for $z \in \mathbb{C} \setminus \mathbb{D}$.

4. Let $f_1(z)$ and $f_2(z)$ be functions holomorphic in a domain D. Assume that both functions satisfy differential equation $f^{(m)} = P(z, f, f^{(1)}, \ldots, f^{(m-1)})$, where P is a polynomial. Show that if $f_1^{(k)}(z_0) = f_2^{(k)}(z_0)$ for some $z_0 \in D$ and all $k \in \{0, 1 \ldots, m-1\}$, then $f_1 \equiv f_2$ in D.

5. Find $\displaystyle\int_{|z|=2} \frac{z^n dz}{\sqrt{1+z^2}}$.

6. Let $f(z)$ be a univalent function in $\mathbb{D}$ such that $f(0) = 0$ and $D := f(\mathbb{D})$ is symmetric with respect to a line L passing through the origin. Show that $f^{-1}(L \cap D)$ is a diameter of $\mathbb{D}$.

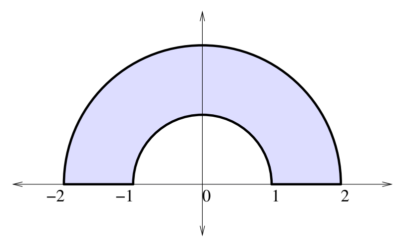
QUALIFYING EXAM
AUGUST 2016
MATH 530 - Prof. Pérez

(1) Let $a > 0$. Find the shortest and longest distances from the origin to the curve

$$\left|z + \tfrac{1}{z}\right| = a.$$

(2) Let C be the (positively oriented) boundary of the half-ring pictured. Find the integral

$$\int_C \frac{z}{\overline{z}}\,\mathrm{d}z.$$



(3) Show that for any radius $r > 0$ there exists $N \geq 1$ such that for all $n \geq N$ the polynomial

$$1 + z + \tfrac{z^2}{2!} + \ldots + \tfrac{z^n}{n!}$$

has no roots within the disk of radius r centered at the origin.

(4) Find a one-to-one conformal mapping of the upper half plane without the vertical ray $[\mathrm{i}, \mathrm{i}\infty)$, onto the unit disc. (The answer can be given as a sequence of maps)

(5) Let f be a holomorphic function defined in a neighborhood of 0, and such that $f(0) = 0$. We say that 0 is a *fixed point* of f. In complex dynamics, two important invariants of such a fixed point are its *multiplier* $\lambda := f'(0)$ and its *fixed point index*

$$\iota := \tfrac{1}{2\pi \mathrm{i}} \int_C \frac{\mathrm{d}z}{z - f(z)},$$

where C is a small, positively oriented curve around 0.

Prove that whenever $\lambda \neq 1$, these invariants are related by the formula

$$\iota = \frac{1}{1-\lambda}.$$

(6) Evaluate the *real* integral

$$\int_0^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4}\,\mathrm{d}x.$$

QUALIFYING EXAM
COMPLEX ANALYSIS (MATH 53000)
AUGUST 2015

In what follows, $\mathbb{D} := \{z : |z| < 1\}$.

1. Find the reflection of the circle $\{|z - a| = r\}$ across the circle $\{|z| = R\}$.

2. Let f be a holomorphic function in $\mathbb{D}$ which is continuous in $\overline{\mathbb{D}}$. Assuming that $|f(z)| < 1$ for all $z \in \overline{\mathbb{D}}$, show that there exists a unique $z_0 \in \mathbb{D}$ such that $f(z_0) = z_0$.

3. Let D be a simply connected domain and $\{f_n\}$ a locally uniformly bounded sequence of univalent functions in D. Assuming that $f_n^{(k)}(z_0) \to w_k$ as $n \to \infty$ for some $z_0 \in D$ and all $k \geqslant 0$ and that w_k are not all zero for $k \geqslant 1$, show that $\{f_n\}$ converges locally uniformly to a univalent function in D.

4. Let f be a holomorphic function in $\mathbb{D}$ except for a simple pole at $z_0 \neq 0$. Show that $a_n/a_{n+1} \to z_0$ as $n \to \infty$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

5. Compute $\displaystyle\int_0^{\pi} \left(\frac{\sin(nx)}{\sin(x)} \right) dx$.

6. Show that $\displaystyle\frac{1}{1-z} = \prod_{n=0}^{\infty} \left(1 + z^{2^n}\right), \quad z \in \mathbb{D}$.

Qualifying Exam — Complex Analysis (Math 53000)

January 2014

V. Tarasov

1. Describe and sketch the set $\operatorname{Re}(\tan z) > 0$.

2. The region D is the intersection of the disks $|z-1| < \sqrt{2}$ and $|z+1| < \sqrt{2}$.
Construct a conformal mapping (one-to-one) from D to the unit disk $|z| < 1$.
First represent the mapping as a composition of elementary mappings, and then simplify the final formula as much as possible.

3. Let $f(z)$ be an entire function that does not take negative real values.
Show that $f(z)$ is a constant function.

4. The function $f(z)$ is analytic in the unit disk $|z| < 1$ and maps it surjectively to the upper half plane $\operatorname{Re} z > 0$. Find the radius of convergence of the Taylor series $\sum_{k=0}^{\infty} a_k z^k$ of the function $f(z)$.

5. a) How many solutions does the equation $2z^2 = \cos z$ have in the unit disk $|z| < 1$?
b) Show that all solutions of the equation $2z^2 = \cos z$ that lie in the unit disk $|z| < 1$ are real.

6. Evaluate $\displaystyle\int_0^\infty \frac{\sin(\pi x)}{x - x^3}\, dx$. Explain each step.

Notation: unit disk: $\mathbb{D} = \{z : |z| < 1\}$ and right half plane: $H_+ = \{z : \text{Re}(z) > 0\}$.

1. (a) Give a careful statement of the Schwarz Lemma, including information about the derivative and equality in the inequalities, which begins: *If* $\varphi : \mathbb{D} \mapsto \mathbb{D}$ *is an analytic function mapping the unit disk into itself such that* $\varphi(0) = 0$, *then* $\cdots$

   (b) Let f be an analytic function mapping the right half plane, H_+, into itself such that $f(1) = 1$. Prove that for w in H_+,

   $$\left|\frac{f(w)-1}{f(w)+1}\right| \leq \left|\frac{w-1}{w+1}\right|$$

   (c) What can you say about f if there is $w_0 \neq 1$ for which equality holds in this inequality?

2. Find the number of roots of the polynomial $p(z) = z^7 - 8z^2 + 2$ in the annulus $\{z : 1 < |z| < 2\}$. Prove that your answer is correct, citing appropriate theorems and justifying their hypotheses.

3. Let $\mathcal{S} = \{z : \text{Re}(z) > 0 \text{ and } |\text{Im}(z)| < \frac{3\pi}{2}\}$ and let $g(z) = e^z$ for z in $\mathcal{S}$.

   (a) Describe the set $g(\mathcal{S}) = \{w \in \mathbb{C} : w = g(z) \text{ for some } z \text{ in } \mathcal{S}\}$.

   (b) For any set F, let $\#(F)$ denote the number of elements of F, so $\#(\emptyset) = 0$, $\#(\mathbb{C}) = \infty$, and $\#(\{n : n \text{ is an integer and } |n| \leq 10\}) = 21$.

   For each point w in $g(\mathcal{S})$, find $\#(\{z \in \mathcal{S} : g(z) = w\})$, that is, for each w in $g(\mathcal{S})$, find the number of points of $\mathcal{S}$ that map to w.

4. Let $\Omega_1 = \{z = x + iy : -3 < x < 2 \text{ and } y = 0\} \cup \{z = x + iy : x < 2 \text{ and } y \neq 0\}$
   and $\Omega_2 = \{z = x + iy : -3 < x < 2 \text{ and } y = 0\} \cup \{z = x + iy : x > -3 \text{ and } y \neq 0\}$.
   The function ζ is analytic on Ω_1 and has a non-removable singularity at -3. Similarly, the function η is analytic on Ω_2 and has a non-removable singularity at 2.

   Finally, these functions satisfy $\zeta(-1) = \eta(-1)$ and the derivatives $\zeta^{(k)}(-1) = \eta^{(k)}(-1)$ for all positive integers k, where $h^{(1)}(z) = h'(z)$, $h^{(2)}(z) = h''(z)$, etc.

   (a) Find the coefficients a_k (in terms of information about ζ) so that the power series for ζ centered at $c = -1$ is $\sum_{k=0}^{\infty} a_k(z+1)^k$.

   (b) What is the radius of convergence of the power series for ζ centered at $a = -1$? Explain why!

   (c) Prove that $\zeta(z) = \eta(z)$ for all z in $\Omega_1 \cap \Omega_2 = \{z = x + iy : -3 < x < 2\}$.

5. Find the Laurent series for $h(z) = \dfrac{z}{z^2 + 2z - 3}$ that converges on the open annulus $\{z : 1 < |z| < 3\}$.

6. Use contour integration to compute $\displaystyle\int_0^\infty \frac{x^{1/3}}{1 + x^2}\, dx$.

   Justify your calculation by explicitly describing the contour(s), the substitutions used in each piece of the contour(s), and the estimates necessary for drawing your conclusion. (You should express your answer in a form that is obviously a real number!)

Math 53000 – Qualifying Examination – January 2013

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

1. Find the number of zeros of the function

$$f(z) = 2013z^8 + 1000 + e^{(3+4i)z}$$

in $\mathbb{D}$.

2. Evaluate $$\int_0^\infty \frac{\cos x}{x^2+4}\,dx\ .$$

3. Find a one-to-one analytic map from

$$\{z \in \mathbb{C} : |z| < 1,\ \mathrm{Re}(z) > 0,\ \mathrm{Im}(z) > 0\} \setminus \{a(1+i) : a \in (0, 1/2]\}$$

onto $\mathbb{H}$. You may represent your map as a finite composition of maps.

4. Find all analytic maps $f : \mathbb{H} \to \mathbb{H}$ for which $f(i) = i$ and $f'(i) = -1$.

5. Consider the following statements:

(a) If $f : \mathbb{C} \to \mathbb{C}$ is an analytic function then the function $g : \mathbb{C} \to \mathbb{C}$, given by $g(z) = f(\bar{z})$, is also analytic.

(b) If $u : \mathbb{C} \to \mathbb{R}$ is a harmonic function then the function $v : \mathbb{C} \to \mathbb{R}$, given by $v(z) = u(\bar{z})$, is also harmonic.

For each statement, decide whether it is true or false. If it is true, prove it; if it is false, give a counterexample.

6. Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function such that if $|z| = 1/2$ then $f(z) \in \mathbb{R}$. Prove that f is constant.

Qualifying Exam — Complex Analysis (Math 53000)

August 2013

V. Tarasov

1. Describe and sketch the set $\operatorname{Im}(z^3 + z^{-3}) < 0$.

2. The region $D = \{ z \in \mathbb{C} \,:\, |z-1| > 1\,,\ |z+1| > 1 \}$ is the complement of the union of two disks. Construct a conformal mapping (one-to-one) from D to the unit disk $|z| < 1$. First represent the mapping as a composition of elementary mappings, and then simplify the final formula as much as possible.

3. Find all entire functions $f(z)$ such that $|f(z)| = 1$ for $|z| = 1$.
Hint: *Show first that* $f(z)$ *is a polynomial.*

4. The series $\sum\limits_{k=0}^{\infty} a_k z^k$ converges in the unit disk $|z| < 1$ and defines a function mapping the unit disk into itself. Show that $|a_0|^2 + |a_1| \leqslant 1$.

5. Show that the equation $z = \tan(\pi z)$ has exactly three solutions in the strip $|\operatorname{Re} z| < 1$.
Hint: *Keep in mind the proof of Rouche's theorem.*

6. Evaluate $\displaystyle\int\limits_0^{\infty} \frac{\cos(\pi x/2)}{1 - x^4}\, dx$. Explain each step.

Math 53000 – Qualifying Examination – January 2012

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

1. Find the number of zeros of the polynomial

$$P(z) = z^{2012} + z^{20} + z^{12} + i$$

in the second quadrant.

2. Evaluate the integral

$$\int_0^\infty \frac{\sin x \, dx}{x^3 + x} \, .$$

3. Find all entire functions f such that

$$|f(z)| < |z| + 2012$$

for all $z \in \mathbb{C}$.

4. Find a conformal map from $\mathbb{D}$ onto $\mathbb{D} \setminus \left[-\frac{1}{2}, 1\right)$. You may represent your map as a finite composition of maps.

5. Let P be a polynomial with real coefficients. Prove that there is a harmonic function $h : \mathbb{C} \to \mathbb{R}$ such that $h(iy) = P(y)$ for all $y \in \mathbb{R}$.

6. Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function such that $f(0) = 0$ and

$$\sup_{z \in \mathbb{D}} |f(z)| < 1.$$

Define by induction: $z_1 = \frac{1}{2}$, $z_{n+1} = f(z_n)$ for $n = 1, 2, \ldots$. Prove that $\lim_{n \to \infty} z_n = 0$.

Qualifying Exam — Complex Analysis (Math 53000)

January 2011

V. Tarasov

1. Describe and sketch the set $\operatorname{Re}(\tan^2 z) > 0$.

2. Find all analytic functions $f(z)$ defined in the upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ such that $|f(z)| < 1$ for all z, $f(i) = 0$ and $f'(i) = 1/2$.

3. Let a polynomial $P(z) = 1 + a_1 z + \ldots + a_n z^n$, $n \geq 1$, be such that $|P(z)| \geq 1$ for all z on the circle $|z| = 1$. Show that $P(z)$ has a root in the disk $|z| < 1$.

4. Find all zeros and singularities of the function $\dfrac{e^{1/z} \sin z}{1 - \cos z}$ including infinity. Classify the singularities. For each zero and pole, find the order and compute the residue at the pole.

5. Evaluate $\displaystyle\int_{|z|=1} \frac{dz}{\sin(4/z)}$ for the positive sense of the circle.

6. Let $f(z)$ be the branch of $\log z$ such that $0 \leqslant \operatorname{Im} f(z) < 2\pi$. Write the Taylor series for $f(z)$ at $z = 1 - i$. Determine the radius of convergence of the Taylor series. Express the sum of the Taylor series series in terms of $f(z)$.

7. Evaluate $\displaystyle\int_0^\infty \frac{1 - \cos x}{x^2}\, dx$.

Math 53000 – Qualifying Examination – August 2011

1. Find the number of zeros of the polynomial

$$P(z) = z^{2011} - 1000z^{1000} - 2011$$

in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

2. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 14x + 113}.$$

Explain all steps.

3. Let $D = \{z : |z| < 1\}$ be the unit disk. Does there exist an analytic function $f : D \to \mathbb{C}$ such that

$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^3}$$

for $n = 2, 3, 4, \ldots$? Give an example of such a function or prove that it does not exist.

4. Find a conformal map from the set

$$\{z \in \mathbb{C} : |z| > 1\} \setminus (-\infty, -1)$$

onto the set

$$\mathbb{C} \setminus (-\infty, 0].$$

You may represent your map as a finite composition of maps.

5. Find all singular points of the function

$$f(z) = \frac{(\sin(z^3))^7}{z^{21}}$$

and classify them.

6. Find all harmonic functions $f : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ that are constant on every circle centered at 0.

Qualifying Examination – Math 53000 – 1/4/2010

Notation:
$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\quad \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$, $\quad \overline{\mathbb{H}} = \{z \in \mathbb{C} : \operatorname{Im} z \ge 0\}$.

1. Evaluate $\displaystyle\int_0^\infty \frac{\cos x}{(x^2-1)(x^2-9)}\, dx.$

2. Does there exist an analytic function f, defined in a neighborhood of $\overline{\mathbb{D}}$, such that $f(\overline{\mathbb{D}}) = \overline{\mathbb{H}}$? Give an example or prove that it does not exist.

3. Prove that there is no entire function f such that $|f(z)| > |z|$ for all $z \in \mathbb{C}$.

4. Find all singular points in $\mathbb{C}$ and residues at those points of the function

$$f(z) = \frac{z^4}{(z^2-4)^3} \ .$$

5. Write the Laurent series of the function

$$f(z) = \frac{z^{2010}}{(z-1)(z+2)}$$

in the annulus $1 < |z| < 2$.

6. Find all analytic functions $f : \mathbb{D} \to \mathbb{D}$ such that $f(0) = 1/3$ and $f(1/3) = 0$.

Qualifying Exam — Complex Analysis (Math 53000)

August 2010

1. Describe and sketch the set $\operatorname{Im} \dfrac{(z-1)^3}{(z+1)^3} > 0$.

2. Find all analytic functions $f(z)$ defined in the unit disk $\{z \in \mathbb{C} \,:\, |z| < 1\}$ such that $\operatorname{Im} f(z) > 0$ for all z, $f(0) = i$ and $f(1/3) = 2i$.

3. Let $f(z)$ be an analytic function in the disk $\{z \in \mathbb{C} \,:\, |z| < R\}$, where $R > 1$.

Prove that $\displaystyle\int_{|z|=1} \overline{f(z)}\, dz = 2\pi i\, \overline{f'(0)}$. The integral is for the positive sense of the circle.

4. Let $f(z) = \dfrac{\log(z^2+1)}{(z+2)^2}$, where $-\pi \leqslant \operatorname{Im} \log(z^2+1) < \pi$. Write the Laurent series for $f(z)$ around $z = -2$. Determine if the obtained series is convergent at $z = 1$, $z = 0$, $z = -1$, and find the sum of the series at the points where the series is convergent.

5. Define by induction: $a_0 = 1/2$, $a_1 = 1/3$, and $a_n = a_{n-1}\, a_{n-2}$ for $n \geqslant 2$. Prove that the series $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is convergent for all z. Show that there exists a sequence $z_1, z_2, \ldots$ such that $\lim\limits_{n\to\infty} z_n = \infty$ and $\lim\limits_{n\to\infty} f(z_n) = 0$.

6. Evaluate $\displaystyle\int_0^\infty \frac{x\sqrt{x}}{x^3+1}\, dx$ using residues. Simplify the answer as much as possible.

Qualifying Examination – Math 53000 – 8/17/2009

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}, \quad \mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$

1. Evaluate $\displaystyle\int_0^\infty \frac{x^2+1}{x^4+1}\,dx.$

2. How many zeros does the polynomial $z^9 + z^4 + 4$ have in the first quadrant? Explain.

3. Let $\mathcal{F}$ be the set of all analytic functions $f : \mathbb{H} \to \mathbb{H}$ such that $f(i) = 2009\,i$. Find $\displaystyle\sup_{f\in\mathcal{F}} |f'(i)|$.

4. What are the possible types of singularity of the product fg, if $f, g : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ are analytic functions, f has an essential singularity at 0, and:

a. g has an essential singularity at 0?

b. g has a pole at 0?

Explain.

5. Define by induction: $a_0 = 2i$, $a_{n+1} = a_n^2$. Find the radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} a_n z^{n!}$.

6. Find the image of $\mathbb{D}$ under the map $f(z) = z^2 - \dfrac{1}{z^2}$.

Qualifying Examination (Complex Analysis)

IUPUI, August 2008 – Rodrigo A. Pérez

1. Find a on-to-one conformal mapping from the upper half-plane without the segment $(0, i]$ onto the unit disk (the answer can be given as a series of maps).

2. (a) Determine the number of points z in the unit disk such that $e^z = 3z^4$.
   (b) Determine the number of points z in the complex plane such that $e^z = 3z^4$.

3. Use contour integration to compute the integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}.$$

4. Suppose $f(z)$ is an entire fuction such that $|f(z)| \leq A|z|$ for all z, where A is some fixed positive number. Show that $f(z) = az$, where a is a complex constant.

5. Compute the Laurent series representation of $\dfrac{e^z}{(z+1)^2}$ around $z = -1$ and show that it is valid for $0 < |z+1| < \infty$.

6. Let γ be the circle $|z| = 2$ described counterclockwise. Evaluate the integral

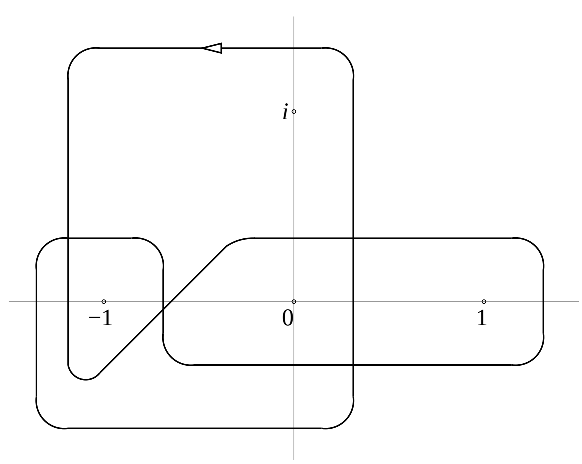
$$\int_\gamma \frac{5z - 2}{z(z-1)}\, dz$$

   (a) by finding the residues at the finite poles,
   (b) by finding the residue at ∞.

7. Compute

$$\frac{1}{2\pi i} \int_\gamma \frac{z^3 + z}{z - z_0}\, dz,$$

   where z_0 stands for each of $-1, 0, 1, i$ and γ is the curve below



8. Let $P(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n. Prove that if $|a_0| < |a_n|$, then P has at least one zero inside the unit disk.

Qualifying Exam in Complex Analysis
IUPUI, August 2007

Problem 1. Find all real a, b such that $a + ib = i^{i^i}$.

Problem 2. Compute

$$\int_0^\infty \frac{x^{1/3}dx}{x^3+1}.$$

Problem 3. Show that

$$\sum_{j=1}^{\infty} \frac{\sin(n\alpha)}{n} = \frac{\pi - \alpha}{2}.$$

Problem 4. Let

$$\Omega = \{z \in \mathbb{C},\ |z| > 1,\ z \notin \mathbb{R}_{<-1},\ z \notin \mathbb{R}_{\geq 2}\}.$$

Find a conformal map which maps the region Ω to the upper half plane.

Problem 5. Let $D = \{z \in \mathbb{C},\ |z| < 1\}$ be the unit disk. Let f_n, $n \in \mathbb{Z}_{>0}$, be a sequence of functions which are holomorphic in D. Assume that $f_n(z) \neq 0$ for all $n > 0$ and $z \in D$. Assume that for any $z \in D$ there exists a limit $\lim_{n\to\infty} f_n(z) = f(z)$ and that the convergence is uniform on any compact subset of the unit disk. Show that if there exists $z_0 \in D$ such that $f(z_0) \neq 0$ then $f(z) \neq 0$ for all $z \in D$.

Problem 6. Let $D = \{z \in \mathbb{C},\ |z| < 1\}$ be the unit disk. Let f, g be functions which are harmonic in D. Assume that $f(z) = g(z)$ for all $z \in \mathbb{R} \bigcap D$. Is it true that $f(z) = g(z)$ for all $z \in D$?