Qualifying Exam — PDE (Math 52300)

January 2017

V. Tarasov

1. Given numbers a, b such that $b > 0$, let $C_{a,b}$ be the following parametrized curve: $x = at + bt^2$, $y = bt^2 + 1$, $z = at + b$, $t > 0$. Find all vector fields $\mathbf{V}(x, y, z)$ in the domain $\{y > 1,\ x + 1 < y + z\}$ such that for any a, b, the curve $C_{a,b}$ is an integral curve of $\mathbf{V}$.

2. Solve the initial value problem in the domain $\{x > 0,\ y > 0\}$:
$xyz_x + 2x^2yz_y + 2(x^2 + y)z = x^2 + y$, and $z = y$ on the curve $x^2y = 1$.

3. Consider the differential equation $z_t - 2z_x(z + 1) = 0$ for $t \geqslant 0$.

a) For the initial condition $z(x, 0) = \dfrac{2x}{x^2 + 1}$, find t_c, the value of t when the shock develops. Sketch the graphs of $z(x, 0)$ and $z(x, t_c)$ as functions of x.
Point out the feature of the graph of $z(x, t_c)$ that indicates the shock developing.
The graphs should be sufficiently accurate. Find $z_x(4, t_c)$.

b) Find the minimum and maximum values of $z(x, t)$ as $-\infty < x < \infty$, $0 \leqslant t < t_c$.

c) Compute the integral $\displaystyle\int_{-\infty}^{\infty} z(x, t_c)\, dx$.

You need not find an analytic formula for the solution $z(x, t)$.

4. Find all solutions of the boundary value problem in the square $\{|x| < 1,\ |y| < 1\}$:

$$\begin{aligned}
\Delta u(x, y) &= 0 && \text{for } |x| < 1,\ |y| < 1, \\
\frac{\partial u}{\partial n}(\pm 1, y) &= 1 && \text{for } |y| < 1, \\
\frac{\partial u}{\partial n}(x, \pm 1) &= -1 && \text{for } |x| < 1.
\end{aligned}$$

5. Find all solutions of the boundary value problem in the unit disk:

$$\begin{aligned}
\Delta u(x, y) &= x^2 + y^2, && \text{for } 0 < x^2 + y^2 < 1, \\
u(x, y) &= 0, && \text{for } x^2 + y^2 = 1.
\end{aligned}$$

6. Let $u(x,y,z)$ be the solution of the boundary value problem in the cube $\Omega = \{\, |x| < 1\,,\ |y| < 1\,,\ |z| < 1\}$:

$$\begin{aligned} \Delta u(x,y,z) &= 0, && \text{for } (x,y,z) \in \Omega\,, \\ u(x,y,z) &= x^2 + y^2 + z^2\,, && \text{for } (x,y,z) \in \partial\Omega\,. \end{aligned}$$

Prove that $0 < u(x,y,z) < 3$ for $(x,y,z) \in \Omega$.

7. Use D'Alembert's method to solve the problem for the wave equation in the quadrant:

$$\begin{aligned} u_{tt}(x,t) &= 4u_{xx}(x,t)\,, && x > 0\,, \quad t > 0\,, \\ \frac{\partial u}{\partial n}(0,t) &= \cos t\,, && \qquad\qquad t > 0\,, \\ u(x,0) &= e^{-x}, && x > 0\,, \\ u_t(x,0) &= 2e^{-x}, && x > 0\,. \end{aligned}$$

Find the limit $\lim\limits_{x\to\infty} u(x,t)$ as a function of t.

8. Let Ω be a bounded normal domain in $\mathbb{R}^3$. Write down a proof that the initial boundary value problem

$$\begin{aligned} u_t(\mathbf{x},t) &= \Delta u(\mathbf{x},t)\,, && \mathbf{x} \in \Omega\,, \\ \frac{\partial u}{\partial n}(\mathbf{x},t) &= 0\,, && \mathbf{x} \in \partial\Omega\,, \\ u(\mathbf{x},0) &= 0\,, && \mathbf{x} \in \Omega\,. \end{aligned}$$

has at most one solution in $C^2(\overline{\Omega} \times \mathbb{R}_{\geqslant 0})$.

Qualifying Exam — PDE (Math 52300)

January 2016

V. Tarasov

1. For any numbers a, b, let $C_{a,b}$ be the curve in $\mathbb{R}^3$ with the following parametrization: $x = a\cos t + b\sin t$, $y = a\sin t - b\cos t$, $z = t$, $-\infty < t < \infty$. Find all vector fields $\mathbf{V}(x,y,z)$ in $\mathbb{R}^3$ such that for any a, b, the curve $C_{a,b}$ is an integral curve of $\mathbf{V}$.

2. Solve the initial value problem in the domain $x > 0$, $y > 0$: $z_x - yzz_y = z^2$, and $z = 1/x$ on the parabola $x = y^2$.

3. Consider the differential equation $(z^2+3)\,z_t - z_x = 0$ for $t \geqslant 0$.

a) For the initial condition $z(x,0) = \begin{cases} 3, & x > 3, \\ x, & |x| \leqslant 3 \\ -3, & x < -3 \end{cases}$, find t_c, the value of t when the shock develops. Sketch the graphs of $z(x,0)$ and $z(x,t_c)$ as functions of x. Point out the feature of the graph of $z(x,t_c)$ that indicates the shock developing. The graphs should be sufficiently accurate.

b) Find the minimum and maximum values of $z(x,t_c)$ as $-\infty < x < \infty$.

b) Compute the integral $\displaystyle\int_{-16}^{16} z(x,t_c)\,dx$.

You are not required to find the function $z(x,t)$ analytically.

4. Let $u \in C^2(\mathbb{R}^2)$ be the solution of the boundary value problem:

$$\Delta u(x,y) = 0 \qquad \text{for } x^4 + y^4 < 1,$$
$$u(x,y) = x^2 + y^2 \qquad \text{for } x^4 + y^4 = 1.$$

Show that $1 < u(x,y) < \sqrt{2}$ if $x^4 + y^4 < 1$.

5. Let $\mathbb{S}^1$ be the unit circle $\{x^2+y^2=1\} \subset \mathbb{R}^2$. Find a function $u(x,y) \in C^2(\mathbb{R}^2 \setminus \mathbb{S}^1) \cap C^1(\mathbb{R}^2)$ such that

$$\Delta u(x,y) = x \qquad \text{for } x^2+y^2 < 1,$$
$$\Delta u(x,y) = 0 \qquad \text{for } x^2+y^2 > 1,$$
$$u(x,y) \to 0 \qquad \text{as } x^2+y^2 \to \infty.$$

6. Let Ω be a bounded normal domain in $\mathbb{R}^3$. Suppose $u \in C^2(\overline{\Omega} \times \mathbb{R}_{\geqslant 0})$ is such that

$$\begin{aligned} u_t(\mathbf{x}, t) &= \Delta u(\mathbf{x}, t)\,, \qquad && \mathbf{x} \in \Omega\,, \\ u(\mathbf{x}, t) &= 0\,, && \mathbf{x} \in \partial\Omega\,. \end{aligned}$$

a) Let $f \in \mathbb{C}^2(\mathbb{R})$ be an even function, $f(s) = f(-s)$, such that $f''(s) \geqslant 0$ for all s.
Set $E(t) = \displaystyle\int_\Omega f\big(u(\mathbf{x}, t)\big)\, d^3\mathbf{x}$. Show that $\dfrac{dE}{dt} \leqslant 0$.

b) Let $M(t) = \max_{\mathbf{x} \in \overline{\Omega}} |u(\mathbf{x}, t)|$. Use part a) to show that $M(t) \leqslant M(0)$ for all $t \geqslant 0$.
Hint. Choose an appropriate function $f(s)$.

Qualifying Exam — PDE (Math 52300)

August 2016

V. Tarasov

1. For any numbers a, b, let $C_{a,b}$ be the curve in the domain $x > 1$ with the following parametrization: $x = at + 1$, $y = t + b$, $z = at + b$. Find all vector fields $\mathbf{V}(x, y, z)$ in the domain $x > 1$ such that for any a, b, the curve $C_{a,b}$ is an integral curve of $\mathbf{V}$.

2. Solve the initial value problem in the domain $x > 0$: $xy^3 z_x + z^2 z_y = -y^3 z$, and $z = y^2$ on the hyperbola $xy = 1$.

3. Consider the differential equation $(z+1)\, z_t + z_x = 0$ for $t \geqslant 0$.

a) For the initial condition $z(x,0) = \dfrac{1}{x^2+1}$, find t_c, the value of t when the shock develops. Sketch the graphs of $z(x,0)$ and $z(x,t_c)$ as functions of x.
Point out the feature of the graph of $z(x,t_c)$ that indicates the shock developing.
The graphs should be sufficiently accurate.

b) Find the minimum and maximum values of $z(x,t)$ as $-\infty < x < \infty$, $0 \leqslant t < t_c$.

c) Compute the integral $\displaystyle\int_{-\infty}^{\infty} z(x,t_c)\, dx$.

You do not need to find an analytic formula for the solution $z(x,t)$.

4. Let $\Omega \subset \mathbb{R}^2$ be the domain $x > 0$, $y > 0$, $x + y < 1$.
Find all solutions of the boundary value problem

$$\begin{aligned} \Delta u(x,y) &= 0 && \text{for } (x,y) \in \Omega, \\ u(x,y) &= y^3 - 3x^2 y && \text{for } (x,y) \in \partial\Omega. \end{aligned}$$

5. Use separation of variables to solve the boundary value problem in the strip:

$$\begin{aligned} &\Delta u(x,y) = 3u && \text{for } 0 < x < \pi/2,\ y > 0, \\ &\frac{\partial u}{\partial n}(0,y) = 0, \quad u(\pi/2, y) = 0, && \text{for } y > 0, \\ &u(x,0) = \cos^3 x, && \text{for } 0 < x < \pi/2, \\ &u(x,y) \text{ is bounded} && \text{as } y \to \infty. \end{aligned}$$

6. Use D'Alembert's method to solve the problem for the wave equation in the quadrant:

$$\begin{aligned} 4u_{tt}(x,t) &= u_{xx}(x,t)\,, && x>0\,, \quad t>0\,, \\ u(0,t) &= \sin t\,, && \qquad\qquad t>0\,, \\ u(x,0) &= 0\,, && x>0\,, \\ u_t(x,0) &= 0\,, && x>0\,. \end{aligned}$$

Find the limit $\lim\limits_{x\to\infty} u(x,t)$ as a function of t.

7. Let Ω be a bounded normal domain in $\mathbb{R}^3$ and $f \in C(\overline{\Omega} \times \mathbb{R}_{\geqslant 0})$.
Prove that the initial boundary value problem

$$\begin{aligned} u_t(\mathbf{x},t) &= \Delta u(\mathbf{x},t) + f(\mathbf{x})\,, && \mathbf{x} \in \Omega\,, \\ u(\mathbf{x},t) &= 0\,, && \mathbf{x} \in \partial\Omega\,, \\ u(\mathbf{x},0) &= 0\,, && \mathbf{x} \in \Omega\,. \end{aligned}$$

has at most one solution in $C^2(\overline{\Omega} \times \mathbb{R}_{\geqslant 0})$.

Qualifying Exam — PDE (Math 52300)

January 2014

V. Tarasov

1. a) Find the general solution of the equation $(x-2y)\,z_x - y\,z_y = x - y - z$.

b) Find the solution in the domain $\{(x,y)\in\mathbb{R}^2 \mid x<y\,,\ y>0\}$ that satisfy the initial condition $z(0,y)=y^2$.

2. Solve the initial value problem for $t \geqslant 0$:

$$z_t + z^2 z_x = 0\,, \qquad z(x,0) = \begin{cases} -1\,, & x \leqslant -1\,, \\ x\,, & |x| \leqslant 1\,, \\ 1\,, & x \geqslant 1\,. \end{cases}$$

Write a formula for the solution $z(x,t)$ that is explicitly continuous at $t=0$.
Find t_c, the value of t when the shock develops.
Sketch the graph of the solution $z(x,t)$ at $t=0$, $t=\dfrac{t_c}{2}$, $t=t_c$.

3. Use separation of variables to solve the boundary value problem in the square

$$\begin{aligned} \Delta u(x,y) &= 0\,, & & 0<x<\pi/2\,, \quad 0<y<\pi/2\,, \\ u(x,0) &= \sin x\,, & & u_y(x,\pi/2)=0\,, \\ u(0,y) &= \sin y\,, & & u_x(\pi/2,y)=0\,. \end{aligned}$$

4. Let Ω be a bounded normal domain in $\mathbb{R}^3$. Prove that the boundary value problem $\Delta u = 1$ in Ω, and $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega$, has no solution in $C^2(\overline{\Omega})$.

5. Use separation of variables to solve the initial value problem on the line $-\infty < x < \infty$:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) - u(x,t)\,, \\ u(x,0) = 2\cos^2 x\,. \end{cases}$$

6. Let Ω be a bounded normal domain in $\mathbb{R}^2$. Suppose $u \in C^2(\overline{\Omega}\times\mathbb{R})$ is such that

$$\begin{aligned} u_{tt}(\mathbf{x},t) &= \Delta u(\mathbf{x},t) - e^{u(\mathbf{x},t)}\,, & & \mathbf{x}\in\Omega\,, \\ u(\mathbf{x},t) &= 0\,, & & \mathbf{x}\in\partial\Omega\,. \end{aligned}$$

Let $E(t) = \displaystyle\int_\Omega \Bigl(\bigl(u_t(\mathbf{x},t)\bigr)^2 + 2e^{u(\mathbf{x},t)} + |\operatorname{grad} u(\mathbf{x},t)|^2 \Bigr)\, d^2\mathbf{x}$. Show that $\dfrac{dE}{dt} = 0$.

Qualifying Exam — PDE (Math 52300)

August 2014

V. Tarasov

1. a) Find the general solution of the equation $xz_x - yz_y = x^2/z$ in the domain $z < 0$.

b) Find the solution of the equation that satisfy the initial condition $z(x, x) = -1$.

2. Consider the initial value problem for $t \geqslant 0$: $z_t - z^2 z_x = 0$, $z(x, 0) = e^{-x^2}$.

a) Find t_c, the value of t when the shock develops.

b) Sketch the graph of the solution $z(x, t)$ at $t = 0$ and at $t = t_c$. The graphs should be sufficiently accurate. At the graph of $z(x, t_c)$, point out the feature that indicate the developing of the shock.

c) Find the maximum value of $z(x, t)$ as $-\infty < x < \infty$, $0 \leqslant t < t_c$.

(*An analytic formula for the solution* $z(x, t)$ *cannot be found in elementary functions.*)

3. Consider the boundary value problem in the unit disk:

$$\Delta u(x, y) = 1, \qquad x^2 + y^2 < 1,$$

$$\frac{\partial u}{\partial n}(x, y) = \lambda, \qquad x^2 + y^2 = 1,$$

where λ is a real number.

a) Show that if $u_1(x, y)$ and $u_2(x, y)$ are two solutions in $C^2(\overline{\Omega})$ of the boundary value problem, then $u_1 - u_2$ is a constant function.

b) Find all values of λ such that the boundary value problem has a solution in $C^2(\overline{\Omega})$.

c) For the values of λ such that the boundary value problem has a solution in $C^2(\overline{\Omega})$, solve the boundary value problem, and prove that you have found all solutions of the boundary value problem.

4. Use separation of variables to solve the initial value problem on the line $-\infty < x < \infty$:

$$\begin{cases} u_{tt}(x, t) = u_{xx}(x, t) + u(x, t), \\ u(x, 0) = 2\sin^2 x, \\ u_t(x, 0) = \sin x. \end{cases}$$

5. Let Ω be a bounded normal domain in $\mathbb{R}^3$. Suppose $u \in C^2(\overline{\Omega} \times \mathbb{R}_{\geqslant 0})$ is such that

$$\begin{aligned} u_t(\mathbf{x}, t) &= \Delta u(\mathbf{x}, t) - \sin\big(u(\mathbf{x}, t)\big), & \mathbf{x} \in \Omega, \\ u(\mathbf{x}, t) &= 0, & \mathbf{x} \in \partial\Omega. \end{aligned} \qquad (1)$$

a) Let $E(t) = \displaystyle\int_\Omega \Big(|\operatorname{grad} u(\mathbf{x}, t)|^2 - \cos\big(u_t(\mathbf{x}, t)\big)\Big)\, d^3\mathbf{x}$. Show that $\dfrac{dE}{dt} \leqslant 0$.

b) Use part a) to show that there is a unique function $u \in C^2(\overline{\Omega} \times \mathbb{R}_{\geqslant 0})$ satisfying (1) and such that $u(\mathbf{x}, 0) = 0$.

Qualifying Exam — PDE (Math 52300)

January 2013

V. Tarasov

There are 7 problems in the exam.

1. Solve the initial value problem in the domain $x > 0$:
$2yzz_x - 3x^2zz_y = 2xy$, and $z = -1$ on the curve $y = x^{3/2}$.

2. Consider the initial value problem for $t > 0$:

$$u_t - \frac{1}{6}(u^3 - u)\,u_x = 0\,, \qquad u(x,0) = \begin{cases} 1\,, & x \leqslant -1\,, \\ -2x-1\,, & -1 \leqslant x \leqslant 0\,, \\ x-1\,, & 0 \leqslant x \leqslant 2\,, \\ 1\,, & 2 \leqslant x\,, \end{cases}$$

a) Sketch the graph of the solution $u(x,t)$ at $t = 0$, $t = 1$, $t = 2$.
b) Determine the value of $t > 0$, when a shock develops.
c) Sketch the graph of $u(x,t_0)$, where t_0 is the value of t determined in item b).
The graphs should be sufficiently accurate, but you do not need to write down a formula for $u(x,t)$.

3. Use separation of variables to solve the boundary value problem in the unbounded domain:

$$\begin{aligned} \Delta u(x,y) &= 0\,, && x^2+y^2 > 4\,, \\ u(x,y) &= xy^2\,, && x^2+y^2 = 4\,, \\ u(x,y) &\to 0\,, && (x,y) \to \infty\,. \end{aligned}$$

Write the answer as a function of x, y.

4. Let Ω be a bounded normal domain in $\mathbb{R}^3$. Show that the boundary value problem $\Delta u = u - 1$ in Ω, and $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega$, has a unique solution in $C^2(\overline{\Omega})$.

5. Solve the initial value problem for the wave equation on the line $-\infty < x < \infty$:

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t)\,, \\ u(x,0) = 0\,, \\ u_t(x,0) = \dfrac{1}{x^2+1}\,. \end{cases}$$

Find the limit $\lim\limits_{t\to\infty} u(x,t)$ as a function of x.

6. Use separation of variables to solve the initial boundary value problem on the line $-\infty < x < \infty$:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) - u(x,t)\,, \\ u(x,0) = \cos^2 x\,. \end{cases}$$

7. Let Ω be a bounded normal domain in $\mathbb{R}^2$. Suppose $u \in C^2(\overline{\Omega})$ is such that

$$\begin{aligned} u_t(x,t) &= \Delta u(x,t) - u(x,t)\,, \qquad && x \in \Omega\,, \\ u(x,t) &= 0\,, && x \in \partial\Omega\,. \end{aligned} \tag{1}$$

a) Let $E(t) = \int\limits_{\Omega} \big(u(x,t)\big)^2\, d^2\mathbf{x}$. Show that $\dfrac{dE}{dt} < 0$ if $u(x,t)$ is a nonzero function.

b) Use part a) to show that there is a unique solution $u \in C^2(\overline{\Omega})$ of the initial boundary value problem associated with (1).

Qualifying Exam — PDE (Math 52300)

January 2012

V. Tarasov

1. a) Find the general solution of the equation $yu_x - u_y - zu_z + z = 0$.

b) Solve the initial value problem: $u = 2xz$ on the plane $y = 0$.

2. Solve the initial value problem for $t \geqslant 0$:

$$(z+2)\,z_x - z_t = 0\,, \qquad z(x,0) = \begin{cases} 0\,, & x \leqslant 0\,, \\ x/3\,, & 0 \leqslant x \leqslant 3\,, \\ 1\,, & x \geqslant 3\,. \end{cases}$$

Sketch the graph of the solution $z(x,t)$ at $t=0$, $t=1$, $t=2$.
At what value of t will a shock develop?

3. Use separation of variables to solve the boundary value problem in the halfplane

$$\Delta u(x,y) = 0\,, \qquad y > 0\,,$$
$$u(x,0) = \cos^2 x\,,$$
$$u(x,y) \text{ is bounded}, \quad (x,y) \to \infty\,.$$

4. Let Ω be a bounded normal domain in $\mathbb{R}^3$. Prove that the boundary value problem $\Delta u = \dfrac{1}{u}$ in Ω, and $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega$, has no solution in $C^2(\overline{\Omega})$. Use Green's identities.

5. Solve the initial value problem for the wave equation on the line $-\infty < x < \infty$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}\,, \\ u(x,0) = \dfrac{1}{e^x + 1}\,, \\ u_t(x,0) = \dfrac{e^x}{(e^x+1)^2}\,. \end{cases}$$

Find the limit $\lim\limits_{t\to\infty} u(x,t)$.

6. Let Ω be a bounded normal domain in $\mathbb{R}^2$. Suppose $u \in C^2(\overline{\Omega})$ is such that

$$\begin{aligned} u_{tt}(x,t) &= \Delta u(x,t) - u(x,t)\,, && x \in \Omega\,, \\ u(x,t) &= 0\,, && x \in \partial\Omega\,. \end{aligned} \tag{1}$$

a) Let $E(t) = \displaystyle\int_\Omega \Big(\big(u_t(x,t)\big)^2 + \big(u(x,t)\big)^2 + |\operatorname{grad} u(x,t)|^2\Big)\, d^2\mathbf{x}$. Show that $\dfrac{dE}{dt} = 0$.

b) Use part a) to show that there is a unique solution $u \in C^2(\overline{\Omega})$ of the initial boundary value problem associated with (1).

Qualifying Exam — PDE (Math 52300)

January 2011

V. Tarasov

1. Solve the initial value problem in the domain $x > 0$:
$xyz_x + x^2 z_y = 2xy - yz$, and $z = y$ on the curve $x = 2y$.

2. Solve the initial value problem for $t \geqslant 0$:

$$(2-z)\,z_x + z_t = 0\,, \qquad z(x,0) = \begin{cases} 0\,, & x \leqslant 0\,, \\ x\,, & 0 \leqslant x \leqslant 1\,, \\ 1\,, & x \geqslant 1\,. \end{cases}$$

Sketch the graph of the solution $z(x,t)$ at $t = 0$, $t = 1/3$, $t = 2/3$.
At what value of t will a shock develop?

3. Use separation of variables to solve the boundary value problem in the unbounded domain:

$$\begin{aligned} \Delta u(x,y) &= 0\,, && x^2+y^2 > 4\,, \\ u(x,y) &= xy\,, && x^2+y^2 = 4\,, \\ u(x,y) &\to 0\,, && (x,y) \to \infty\,. \end{aligned}$$

4. Let Ω be a bounded normal domain in $\mathbb{R}^2$. Prove that the boundary value problem $\Delta u = u^3$ in Ω, and $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega$, has a unique solution $u(x,y) = 0$ in $C^2(\overline{\Omega})$.

5. Solve the initial value problem for the heat equation $\begin{cases} 2\,\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}\,, & t > 0\,, \\ u(x,0) = \sin^2 x\,. \end{cases}$
Find the limit $\lim\limits_{t\to\infty} u(x,t)$.

6. Let Ω be a bounded normal domain in $\mathbb{R}^3$. Suppose $u \in C^2(\overline{\Omega})$ is such that

$$\begin{aligned} u_{tt}(x,t) &= \Delta u(x,t)\,, && x \in \Omega\,, \\ u(x,t) &= 0\,, && x \in \partial\Omega\,. \end{aligned} \tag{1}$$

a) Let $E(t) = \displaystyle\int_\Omega \Big(\big(u_t(x,t)\big)^2 + |\operatorname{grad} u(x,t)|^2\Big)\, d^3\mathbf{x}$. Show that $\dfrac{dE}{dt} = 0$.

b) Use part a) to show that there is a unique solution $u \in C^2(\overline{\Omega})$ of the initial boundary value problem associated with (1).

January 6, 2010

QUALIFYING EXAMINATION
January 8 2010
MATH 530 - Prof. Its

1. Solve the following initial value problem.

$$z^2 z_x + x^3 y z_y = x^3 z,$$

with $z = y^2$ on the line $x = 2y$, $y > 0$.

2. Find the solution to the Dirichlet problem in the annulus:

$$\Delta u = 0, \quad \frac{1}{2} < r < 2, \quad -\pi \le \theta \le \pi,$$

$$u\Big(\frac{1}{2}, \theta\Big) = u\Big(2, \theta\Big) = \cos^3\theta, \quad -\pi \le \theta \le \pi.$$

3. Let Ω be a bounded normal domain in $\mathbf{R}^3$. Suppose that $h(x)$ is harmonic in Ω and that

$$h(x) = \frac{1}{4\pi||x-y||}, \quad \text{if} \quad x \in \partial\Omega,$$

where $y \in \Omega$. Let

$$g(x) = -\frac{1}{4\pi||x-y||} + h(x).$$

Evaluate

$$\int_{\partial\Omega} x_1 \frac{\partial g(x)}{\partial n} d\sigma(x).$$

4. Let $\Omega = (-R, R) \times ... \times (-R, R) \subset \mathbf{R}^n$ be the cube in $\mathbf{R}^n$ centered at the origin and with side-lenght $2R$. Suppose that $u \in C^2 \cap C(\bar{\Omega})$ be a solution to the problem

$$\Delta u = -1 \quad \text{in} \quad \Omega,$$

$$u = 0, \quad \text{on} \quad \partial\Omega.$$

Prove that u satisfy the estimate

$$\frac{R^2}{2n} \le u(0) \le \frac{R^2}{2}.$$

5. Find the solution of the following initial-boundary value problem for the wave equation in $\mathbf{R}^2$

$$\begin{cases} \Delta u - u_{tt} = 0, \quad (x,y) \in \Omega, \quad t > 0 \\ u|_{t=0} = 3\sin x \sin 2y \\ u_t|_{t=0} = 5\sin 3x \sin 4y \\ u|_{\partial\Omega} = 0 \end{cases}$$

where Ω is the rectangle,

$$\Omega = \{(x,y) : 0 < x < \pi, \quad 0 < y < \pi\}.$$

6. Suppose a C^2 solution exists to

$$u_t = \Delta u, \quad x \in \Omega, \quad t > 0,$$

$$u = 0, \quad x \in \partial\Omega, \quad t > 0,$$

$$u(x,0) = \phi(x), \quad x \in \Omega,$$

for a normal bounded domain $\Omega \subset \mathbf{R}^n$. Define F by

$$F(t) = \int_\Omega u^2 dx.$$

(a) Show that $\frac{dF}{dt} \le 0$ and hence $0 \le F(t) \le F(0)$.

(b) Use the part (a) to show that there is at most one C^2 solution to the initial-boundary problem.

IUPUI QUALIFYING EXAM MATH 523
WINTER 2008

PAVEL BLEHER AND ALEXANDER ITS

Problem 1. Solve the initial value problem,

$$\begin{cases} z_x + \dfrac{1}{y}\, z_y = 1 + z^2\,, \\ z(x,1) = \tan(2x), \quad |x| < \dfrac{\pi}{4}\,. \end{cases}$$

Problem 2. Solve the initial value problem for $t \geq 0$,

$$\begin{cases} z z_x + z_t = 0\,, \\ z(x,0) = \sin x, \qquad -\infty < x < \infty. \end{cases}$$

Sketch the graph of the solution on the interval $0 \leq x \leq 2\pi$ at $t = 0$, $t = 0.5$, and $t = 1$. At what time t will a shock develop?

Problem 3. Solve the boundary value problem for the Laplace equation in the disk $\Omega = \{(x,y):\ x^2 + y^2 \leq 1\}$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \\ u(x,y)|_{\partial\Omega} = y^2 + xy. \end{cases}$$

Problem 4. Suppose that $u(x,y)$ solves the following boundary value problem in a bounded normal domain Ω on the plane:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = -1, \\ u(x,y)|_{\partial\Omega} = 0. \end{cases}$$

Prove that $u(x,y) > 0$ for $(x,y) \in \Omega \setminus \partial\Omega$.

Problem 5. Solve the initial value problem for the wave equation,

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}\,, \\ u(x,0) = 0, \\ u_t(x,0) = \dfrac{1}{1+x^2}, \qquad -\infty < x < \infty. \end{cases}$$

Find $\lim\limits_{t\to\infty} u(x,t)$.

Problem 6. Solve the initial-boundary value problem for the heat equation on the interval $0 \le x \le \pi$,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \\ u(x,0) = \sin x \cos^2 x, \\ u(0,t) = 0, \\ u(\pi,t) = 0. \end{cases}$$

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QUALIFYING EXAMINATION
AUGUST 2007
MATH 530 - Prof. Its

1. (15) Find the general solution of the first order linear PDE:

$$u_x + u_y + zu_z + u + x^2 = 0$$

2. (15) Separating variables in rectangular coordinate system, solve

$$\Delta u = 0, \quad 0 < x < a, \quad 0 < y < b,$$
$$u|_{x=0} = A \sin \frac{\pi y}{b}, \quad u|_{x=a} = 0$$
$$u|_{y=0} = B \sin \frac{\pi x}{a}, \quad u|_{y=b} = 0$$

3. (15) Evaluate the integral

$$\int_0^{2\pi} \frac{\sin(n\phi)}{a^2 + r^2 - 2ar \sin\phi} d\phi,$$

where n is an integer and $0 < r < a$. (*Hint: Use the Poisson integral formulae for the solution of the Dirichlet problem for a disc.*)

4. (15) Let Ω be a bounded normal domain in $\mathbf{R}^n$ and let $u(x)$ be a solution of the Dirichlet problem

$$\Delta u = -1, \quad x \in \Omega,$$

$$u = 0, \quad x \in \partial\Omega.$$

where does $u(x)$ attain its minimum ?

5. (10) Let Ω be a bounded normal domain in $\mathbf{R}^n$. Prove that the boundary problem,

$$\Delta u = f, \quad x \in \Omega,$$
$$\frac{\partial u}{\partial n} = 1, \quad x \in \partial\Omega,$$

has no solution if $f \in C(\partial\Omega)$ and $f < 0$.

6. (15) Suppose that $D \subset \mathbf{R}^m$ is a bounded open set, and consider the unbounded open set $\Omega = \mathbf{R}^m \setminus \overline{D}$. Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a harmonic function in Ω such that

$$u = 0 \quad \text{on} \quad \partial\Omega \quad \text{and} \quad \lim_{|x|\to\infty} u(x) = 0. \tag{1}$$

prove that it must be $u \equiv 0$ on Ω (Maximum Principle for unbounded domains).

7. (15) Let Ω be a bounded normal domain in $\mathbf{R}^n$. Show that if u satisfies the hyperbolic equation,

$$\Delta u = u_{tt} + \gamma u_t, \quad x \in \Omega, \quad t > 0 \tag{2}$$

and the Dirichlet boundary condition,

$$u = 0, \quad x \in \partial\Omega, \quad t \geq 0, \tag{3}$$

where $\gamma = \gamma(x,t)$ is a nonnegative continuous function defined for $x \in \overline{\Omega},\ t \geq 0$, then

$$\int_\Omega (|\mathrm{grad}\, u|^2 + u_t^2)|_{t=T} dx \leq \int_\Omega (|\mathrm{grad}\, u|^2 + u_t^2)|_{t=0} dx.$$

Using this inequality, prove the uniqueness of the solution of the initial-boundary problem associated with (2), (3).