Math 261 Practice Exam (9/25/09)
Ch. 13 and 14.1, 14.3-14.5

1. Let $\mathbf{r}(t) = <\sqrt{2-t}, (e^t - 1)/t, \ln(t+1)>$. Find:

a) The domain of $\mathbf{r}(t)$.
b) The limit of $\mathbf{r}(t)$ as $t \to 0$.
c) The derivative $\mathbf{r'}(t)$.

Ans: a) $-1 < t < 0$ or $0 < t \leq 2$.
b) $< \sqrt{2}, 1, 0>$
c) $<\frac{-1}{2*\sqrt{2-t}}, \frac{te^t - e^t + 1}{t^2}, \frac{1}{t+1}>$

2. Find parametric equations for the tangent line to the curve $x = 2\sin(t)$, $y = 2\sin(2t)$, $z = 2\sin(3t)$ at the point $(1, \sqrt{3}, 2)$.

Ans. $x = 1+\sqrt{3}t$, $y = \sqrt{3} + 2t$, $z = 2$.

3. Give the general formula for the length of a space curve $\mathbf{r}(t)$, and use it to find the length of the curve $\mathbf{r}(t) = <2t^{3/2}, \cos(2t), \sin(2t)>$ for $0 \leq t \leq 1$.

Ans. For original exponent on t (1/2), answer is $\sqrt{5} + \frac{1}{4}\ln(9 + 4\sqrt{5})$ (use Maple).

For corrected exponent on t (namely 3/2), the answer is $\frac{2}{27}(13^{\frac{3}{2}} - 8)$.

4. The helix $\mathbf{r}_1(t) = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j} + t\,\mathbf{k}$ intersects the curve $\mathbf{r}_2(t) = (1+t)\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$ at the point (1, 0, 0). Find the angle of intersection of these curves. How is this angle defined?

Ans. $\pi/2$

5. Give formulas for the tangential and normal components of acceleration along some trajectory $\mathbf{r}(t)$. Find the tangential and normal components of acceleration of the trajectory $\mathbf{r}(t) = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j} + t\,\mathbf{k}$ at time t.

Ans. Tangential = $a_T = 0$. Normal = $a_N = 1$, so acceleration vector = N.

6. Give the general formula for curvature and use it to find the curvature at time t of the trajectory in problem 5.

Ans. curvature = 1/2 for all t.

7. Find the parametric equations of the osculating circle of the curve in problem 5 at time $t = \pi/2$.

Ans. x = $-\sqrt{2}\cos(u)$, y = $-2\sin(u) - 1$, z = $\sqrt{2}\cos(u) + \frac{\pi}{2}$.

8. Find the equation of the osculating plane of the curve in problem 5 at time $t = \pi/2$.

Ans. $\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}\left(z - \frac{\pi}{2}\right) = 0$.

9. Find the partial derivatives f_x, f_y, f_{xx} , f_{yy} and f_{xy} for the function $f(x, y) = \ln(e^x + e^y)$. Verify that $z = f(x, y)$ is a solution of the equations $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$ and

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2.$$

Ans. $\frac{\partial z}{\partial x} = \frac{e^x}{e^x+e^y}$, $\frac{\partial z}{\partial y} = \frac{e^y}{e^x+e^y}$, $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{e^{x+y}}{(e^x+e^y)^\wedge 2}$, $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -\frac{e^{x+y}}{(e^x+e^y)^\wedge 2}$.

10. Find the equation of the tangent plane to the surface $z = x^2 + 6y^2$ at the point (2, 1, $\sqrt{10}$).

Ans. z = $4x + 12y - 20 + \sqrt{10}$.

11. Find the linear approximation of the function $f(x, y) = \ln(x - 3y)$ at (7, 2) and use it to approximate $f(6.9, 2.06)$.

Ans. L(x,y) = $1(x - 7) - 3(y - 2) + 0$. $L(6.9, 1.06) = -.28$.

12. Give the definition of differentiability for the function $w = f(x, y, z)$ at a point (a, b, c).

Ans. $\Delta w = f\_x(a,b,c)\Delta x + f\_y(a,b,c)\Delta y + f\_z(a,b,c)\Delta z + \varepsilon 1\Delta x + \varepsilon 2\Delta y + \varepsilon 3\Delta z$, etc., etc.

13. Use the chain rule to find $\partial z/\partial s$ and $\partial z/\partial t$ if z = arcsin($x - y$), $x = s^2 + t^2$, $y = 1 - 2st$.

Ans. Let R = $\sqrt{1 - (x - y)^2}$. Then $\partial z/\partial s = \frac{2s+2t}{R} = \partial z/\partial t$. *(Use symmetry in s and t.)*

14. In the first step of the proof of Kepler's first law of planetary motion, explain why the vector **h** = **r** x **v** must be constant. Ans. See p. 881, top of the page.