

Integration

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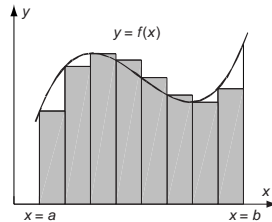
Objective

To illustrate how MATLAB can be used to approximate an area using sums of areas of rectangles.

Narrative

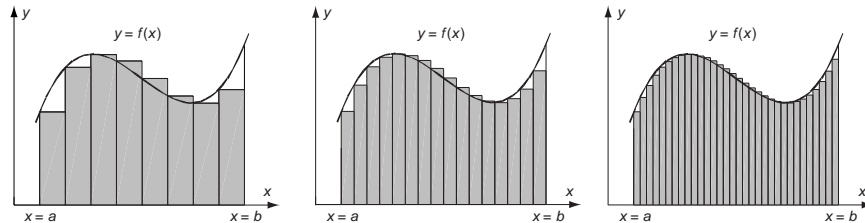
An approximation to the area under the graph of a function $f = f(x)$ for which $f(x) \geq 0$, from $x = a$ to $x = b$ can be obtained by subdividing the interval $[a, b]$ into n non-overlapping subintervals of length $\Delta x = (b - a)/n$ for some integer n , letting $(x_i, f(x_i))$ be a set of sample points on the graph of f , one x_i in each subinterval of the partition of $[a, b]$, and computing the sum

$$\sum_{i=1}^n f(x_i)\Delta x_i.$$



The *exact* area under the graph of f from $x = a$ to $x = b$ is the limit of such sums as n gets large:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x_i.$$



In this project we illustrate these ideas, and that the Fundamental Theorem of Calculus provides a convenient alternative to computing the area directly. The Fundamental Theorem of Calculus states that if $F(x)$ is a function for which $D_x F(x) = f(x)$ then

$$\int_a^b f(x) dx = F(x)\Big|_a^b = F(b) - F(a).$$

In this project we illustrate these ideas using MATLAB's Symbolic Toolbox function `int`. The function `int` can be used in two ways: for any function f ,

`int(f(x), x)` is the antiderivative $F(x) = \int f(x) dx$ of $f(x)$

`int(f(x), x, a, b)` is the definite integral $\int_{x=a}^b f(x) dx$.

Task

In this project we address the problem of finding the area under the graph of $f(x) = x^2 + 1$ from $x = 1$ to $x = 2$.

1. Type the commands in the left-hand column below into MATLAB. The effect of each command is described in the right-hand column for your reference.

<pre>>> % Your name, today's date >> % Integration >> % Task 1 >> clear all, close all</pre>	<p>Clear MATLAB's workspace and close all figure windows.</p>
<pre>>> syms x >> f = inline('x^2+1') >> ezplot(f(x), [0.0,3.0,0.0,10.0], figure(1))</pre>	<p>Let $f(x) = x^2 + 1$. Graph f over the interval $[0, 3]$.</p>

2. Continue by typing the commands in the left-hand column below into MATLAB. These commands compute, for each integer n of the form $n = 2^N$, $N = 1, \dots, 6$, an approximation to

$$\int_a^b f(x) dx = \int_1^2 (x^2 + 1) dx$$

using n subintervals of equal length $\Delta x = (b - a)/n = (2 - 1)/n = 1/n$.

A couple of comments about this code: First, the extra indentation is included only to make the code more readable; it does not affect any of MATLAB's computations. Second, this code uses a "for loop". A "for loop" such as this allows us to perform repeated computations between the initial "for N=1:6" statement and the final "end;" statement. Third, this code uses a "cumulative sum" to compute $\sum_{i=1}^n f(x_i)\Delta x$. For each N , the quantity CumSum is initialized to be 0. Then for each $i = 1, 2, 3, \dots, 2^N$ the summand $f(a + i * dx) * dx$ is added to CumSum.

<pre>>> % Task 2 >> a = 1, b = 2 >> for N=1:6 n = 2^N; dx = (b-a)/n; CumSum = 0; for i=1:n CumSum = CumSum+f(a+i*dx)*dx; end; [n,CumSum] end;</pre>	<p>Let $a = 1$ and $b = 2$. This is the beginning of a new for loop. Let $n = 2^N$ be the number of subintervals. Let dx be the length of each subinterval. Initialize CumSum to 0. This is the beginning of another for loop ... in which we sum the areas ... of the rectangles. Here we report our results. This is the end of the second for loop.</p>
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3. Continue by typing the commands in the left-hand column below into MATLAB. They compute the antiderivative $\int f(x) dx$ of $f(x)$ and $\int_{x=a}^b f(x) dx$.

<pre>>> % Task 3 >> int(f(x),x) >> diff(int(f(x),x)) >> int(f(x),x,a,b)</pre>	<p>Compute the antiderivative $\int f(x) dx$ of $f(x)$. Check our work by computing $D_x(\int f(x) dx)$; it should be $f(x)$. Compute $\int_{x=a}^b f(x) dx$.</p>
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At this time make a hard copy of MATLAB's Command Window and a hard copy of the Figure 1 window. Then:

4. On a separate piece of paper, make by hand a chart whose first column consists of the numbers n of intervals, and whose second column consists of the corresponding sum $\sum_{i=1}^n f(x_i)\Delta x$ (to four significant decimal places). Underneath this table, write the sentence " $\int_{x=1}^2 (x^2 + 1) dx = \underline{\hspace{2cm}}$ " filling in the blank with the correct number (in decimal form, to four significant decimal places).

5. On the hard copy of the figure window, draw and label the coordinate axes, draw the rectangles used to compute $\sum_{i=1}^n f(x_i)\Delta x$ using 4 rectangles, and plot and label the points $(x_i, f(x_i)), i = 1, 2, 3, 4$, used to compute $\sum_{i=1}^n f(x_i)\Delta x$.

Your lab report will consist of the hard copy of MATLAB's Command Window, the paper containing your chart, and the hand-labeled hard copy of the figure window. Staple these all together, the Command Window on top, your chart next, and your figure window last.

Comments

While `for` loops are a standard programming construct, MATLAB offers an efficient alternative: vectorization. Vectorization will be discussed in future courses.