IUPUI

Department of Mathematical Sciences

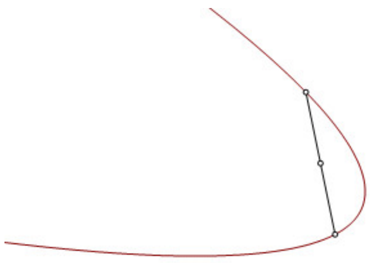
2017 High School Math Contest

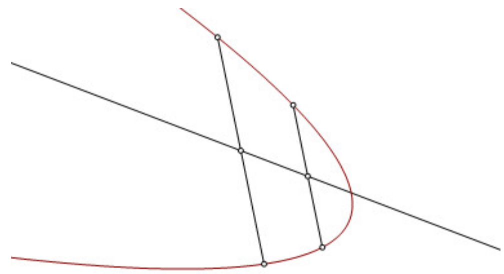
Solutions to problems

Problem 1) Given any arc on a parabola (the part between any two distinct points), use compass and straightedge to construct its focus.

Solution: Begin by constructing an arbitrary chord on the parabola, and construct the midpoint of that chord (Figure 1).

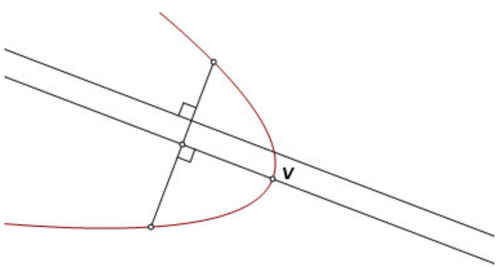
Construct a second chord parallel to the first, and construct its midpoint. The line ℓ through the midpoints is parallel to the axis of symmetry (Figure 2).

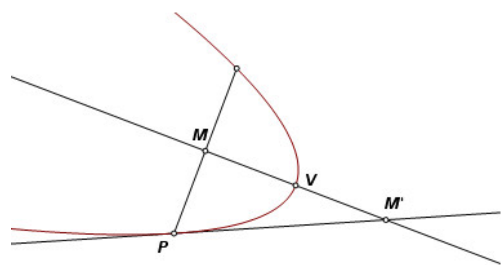




Construct a chord perpendicular to ℓ. The perpendicular bisector of the chord is the axis of symmetry, and it intersects the parabola at the vertex, V (Figure 3).

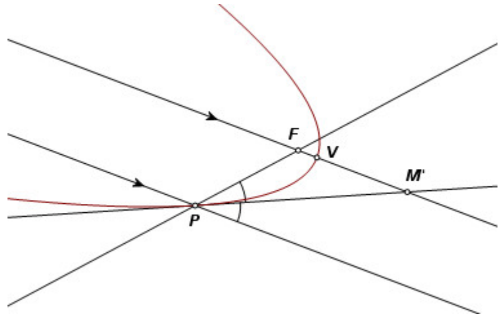
Let P be one endpoint of the orthogonal chord, and let M be the midpoint. Let M' be the image of M reflected through V. That makes line PM' tangent to the parabola at P (Figure 4).





In an ellipse, a tangent line bisects the angle formed by the lines joining the point of tangency to the two foci. A parabola is a limiting case of ellipses, where one focus has been sent to a point at infinity in the direction of the axis. We cannot plot that missing focus, but we know in what direction it lies. With that knowledge we can construct the other focus.

Through point P, construct a line parallel to the axis. Reflect that line across the tangent PM'. The reflected image intersects the axis at the focus F.



Problem 2) Four people, A, B, C, and D, are known to tell the truth only once out of every three times they speak. Each of them does so randomly, independent of what anyone previously said. They all speak; first A, then B, C, and finally D, who says "C negated that B said that A lied" What is the probability that A told the truth in the first place? (We assume that B made a claim about A's statement, and that C made a claim about B's statement.)

Solution: First consider the simplified case of two persons, A and B, in which A makes some statement, and B says something about A's statement. If A speaks the truth, and B lies, it is impossible for B to say "A spoke the truth." Let's call this statement $\mathcal{P}$.

Statement $\mathcal{P}$ is only possible if both A and B lie, or both say the truth. The key observation is that switching one True/False assignment to A or B switches from $\mathcal{P}$ being possible to impossible, or the other way around.

Now consider the original problem. The statement by D is only possible for certain assignments of True/False to A, B, C, and D. Which assignments? It IS possible when all say the truth. Also, it is possible every time we switch two True/False assignments. Therefore, D's statement is possible exactly when an even number of people say the truth! That is, in either of three situations:

- All lie: 1 possible combination.
- Two lie (and two don't): 6 combinations.
- None lie: 1 possible combination.

Now we have to consider the probabilities. Since each person lies two thirds of the time, we should count three times each assignment of True/False to each person (one True assignment and two False assignments). That would make a total of $3^4 = 81$ possibilities, BUT we already know that some assignments are impossible! The correct count is

- All lie: 2 assignments to each of 4 persons give $2^4 = 16$ assignments.
- Two lie: 2 assignments to each of 2 persons in 6 different combinations give $2^2 \cdot 6 = 24$ assignments.
- None lie: 1 more possibility.

This makes a total of $16 + 24 + 1 = 41$ possible scenarios, of which, $3 \cdot 4 + 1 = 13$ involve A saying the truth, so the final answer is $\frac{13}{41}$.

Problem 3) There are n balls of unit mass and of zero radius on a 1-dimensional table of length 1, each currently either motionless or moving in either direction at a velocity identical for all moving balls. Assuming all the collisions are elastic (preserve the momentum and kinetic energy), show that the system is periodic (i.e. there is some time in the future when each billiard ball will again be in its current location and moving at its current velocity).

Solution: How do two balls behave in the event of a collision?

- Conservation of momentum neans that the sum mv_1+mv_2 of momenta *before* impact is the same as the sum mV_1+mV_2 of momenta *after*.
- Conservation of energy neans that the sum $mv_1^2+mv_2^2$ of energies *before* impact is the same as the sum $mV_1^2+mV_2^2$ of energies *after*.

We can cancel the mass m (which is equal for both balls before and after), to get

$$(1) \qquad v_1+v_2=V_1+V_2 \quad , \quad v_1^2+v_2^2=V_1^2+V_2^2 \quad ,$$

where lowercase v's are velocities before, and uppercase V's velocities after impact.

Initially, all velocities are the same, up to a difference in sign, so we can change units to assume that velocities are all ± 1. Out first goal is to show that, after each collision, all velocities remain ± 1:

In order to collide, the left ball must have velocity 1, and the right ball velocity -1. The equations (1) become

$$V_1+V_2=0 \quad , \quad V_1^2+V_2^2=2.$$

Substituting the first equation into the second shows that the only solutions are $(V_1,V_2)=(1,-1)$ and $(-1,1)$, as we claimed.

Notice that only the latter solution makes physical sense so that the left ball remains on the left (and the right ball on the right). But wait! Instead of having balls bounce off each other, we can assume that they pass through each other without altering their motion; this is the meaning of the other solution. If the balls have no identifying marks, this scenario looks indistinguishable from the original one: Two points bouncing off each other, and two points passing through each other *look the same*.

Now that we have modified the problem, the solution is clear. Since all balls have the same speed, and DO NOT INTERACT WITH EACH OTHER, every ball will take the same time to move across the table and return to its original position and direction (after bouncing *twice* off the table edges). This is exactly the statement to be proved.

Problem 4) Find all roots, real and complex, of

$$x^8 - 7x^7 + 14x^6 - 14x^5 + 27x^4 - 14x^3 + 14x^2 - 7x + 1 = 0,$$

and express them in simplest form.

Solution: Notice that the polynomial is symmetric, and that $x = 0$ is NOT a solution. Hence we can divide by x^4 and search for roots of

$$x^4 - 7x^3 + 14x^2 - 14x + 27 - 14x^{-1} + 14x^{-2} - 7x^{-3} + x^{-4} = 0. \quad (2)$$

Let $y = x + x^{-1}$, so that

- $y^2 = (x^2 + x^{-2}) + 2,$
- $y^3 = x^3 + 3x + 3x^{-1} + x^{-3} = (x^3 + x^{-3}) + 3y,$
- $y^4 = x^4 + 4x^2 + 6 + 4x^{-2} + x^{-4} = (x^4 + x^{-4}) + 4y^2 - 2.$

Now we can replace every instance of $x^r + x^{-r}$ in (2) by a function of y (so, for instance, $-7x^3 - 7x^{-3}$ becomes $-7(y^3 - 3y)$):

$$y^4 - 7y^3 + 10y^2 + 7y + 1 = 0.$$

Now we do the same trick, substituting $z = y - y^{-1}$ to obtain

$$z^2 - 7z + 12 = 0$$

The roots of this quadratic polynomial are $z = 3, 4$, which means that

$$y = \frac{3 \pm \sqrt{13}}{2}\ ,\ 2 \pm \sqrt{5},$$

which in turn means that

$$x = \frac{3 + \sqrt{13} \pm \sqrt{6\sqrt{13} + 6}}{4}\ ,\ \frac{3 - \sqrt{13} \pm \mathrm{i}\sqrt{6\sqrt{13} - 6}}{4}\ ,$$

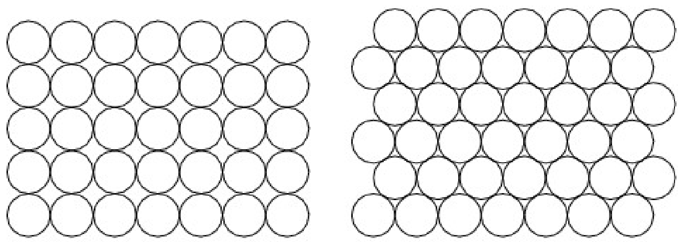
$$\frac{2 + \sqrt{5} \pm \sqrt{4\sqrt{5} + 5}}{2}\ ,\ \frac{2 - \sqrt{5} \pm \sqrt{4\sqrt{5} - 5}}{2}.$$

Team Problem) How many billiard balls of radius 1 can be arranged in a box of dimensions 55x78x101? (Hint: first try to solve the problem for smaller boxes.)

Solution: First, a warning. This is an open-ended problem in that we do not know an *exact* solution. It could be that a very tight packing can be "shaken a bit", shifting some ball positions here and there, to accumulate just enough space for one more ball. However, it is true that the tightest configurations tend to arrange balls in very regular patterns.

We will only compute the number of balls is packings that have balls packed one layer at a time, with the next layer balls fitting in the pockets left by the previous layer.

The two best possibilities are layers of balls arranged in a square pattern (fits less balls in the layer, but has deeper pockets), and in an hexagonal pattern (fits more balls in the layer, but the pockets are not as deep).



square packing *hexagonal packing*

We also need to decide the orientation of the layers; that is, which side of the box is the bottom. Moreover, for the hexagonal pattern, we must decide which side of the bottom is parallel to the hexagonal sides (call this side the *width* of the box bottom). This gives 3 possible packings for the square pattern, and 6 for the hexagonal.

Next, we must determine how many balls fit in a layer. For the square pattern we simply divide by 2 the length and width of the bottom (because the balls have diameter 2), and round down to the nearest integer.

For the hexagonal pattern, we notice that we can fit the same number of balls along the width as we did for the square pattern, but there is extra work to determine the number of rows that fit along the length.

In the final step, we must find how many layers can be stacked in the box. The table below shows the results obtained by one of the participating teams in each of the 9 configurations:

Box base	□ pattern	⬡ pattern (2 orientations)	
101×78	73,150	72,617	72,249
101×55	72,900	72,337	72,505
78×55	73,818	71,108	72,354

The largest value obtained by this team (and a great approximation to the absolute maximum) was $73,818$.