1. Problem 1

The Greek architect Iupuius is designing the main entrance for his masterpiece: The Speedway Neuromath Library. He wants to have 14 columns aligned in a row to hold 7 arches. Each arch will be supported by 2 columns, but the columns do not need to be contiguous. Instead, the arches may extend on top of each other as in the examples below. Iupuius wants to draw all possible arch configurations to decide which looks nicer. His only constraint is to avoid an arch that covers all others; thus, the example on the bottom would be ruled out. How many arch configurations are there? How many if he decides to use 20 columns to hold 10 arches?

Let’s use the symbol $C_n$ to denote the number of configurations with exactly $n$ arches (i.e., with $2n$ columns), but WITHOUT the last restriction about one arch covering all others.

The first few values of $C_n$ are easy to find by trial and error:

\[
C_1 = 1: \begin{array}{|c|} \hline \\
\end{array}
\]

\[
C_2 = 2: \begin{array}{|c|} \hline \\
\end{array}, \begin{array}{|c|} \hline \\
\end{array}
\]

\[
C_3 = 5: \begin{array}{|c|} \hline \\
\end{array}, \begin{array}{|c|} \hline \\
\end{array}, \begin{array}{|c|} \hline \\
\end{array}, \begin{array}{|c|} \hline \\
\end{array}, \begin{array}{|c|} \hline \\
\end{array}
\]

These are the "Catalan numbers"; a famous sequence that shows up in many applications. To find a recursive formula for $C_n$ notice that the left-most arch $L$ in any $n$-arch configuration splits the picture into two shorter configurations: one (possibly empty) covered by $L$, and another (possibly empty) outside and to the right of $L$. For example, in $\begin{array}{|c|} \hline \end{array}$, the left-most arch covers $\begin{array}{|c|} \hline \end{array}$, and leaves $\begin{array}{|c|} \hline \end{array}$ on the right.
The arch $L$ can cover any number of smaller arches, from 0 to $n - 1$, leaving to the right a complementary number of arches, from $n - 1$ down to 0. It follows that the total number of different arch configurations will be

$$C_n = C_0 \cdot C_{n-1} + C_1 \cdot C_{n-2} + \cdots + C_{n-1} \cdot C_0$$

(here, $C_0$ denotes a ”configuration with 0 arches”, and we count it as equal to 1).

For instance, $C_3 = C_0 \cdot C_2 + C_1 \cdot C_1 + C_2 \cdot C_0 = 2 + 1 + 2 = 5$ (compare the examples above). Now we can use our formula to find $C_7$:

$$C_4 = C_0 \cdot C_3 + C_1 \cdot C_2 + C_2 \cdot C_1 + C_3 \cdot C_0$$
$$= 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 = 14.$$

$$C_5 = C_0 \cdot C_4 + C_1 \cdot C_3 + C_2 \cdot C_2 + C_3 \cdot C_1 + C_4 \cdot C_0$$
$$= 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 = 42.$$

$$C_6 = C_0 \cdot C_5 + C_1 \cdot C_4 + C_2 \cdot C_3 + C_3 \cdot C_2 + C_4 \cdot C_1 + C_5 \cdot C_0$$
$$= 1 \cdot 42 + 1 \cdot 14 + 2 \cdot 5 + 5 \cdot 2 + 14 \cdot 1 + 42 \cdot 1 = 132.$$

$$C_7 = C_0 \cdot C_6 + C_1 \cdot C_5 + C_2 \cdot C_4 + C_3 \cdot C_3 + C_4 \cdot C_2 + C_5 \cdot C_1 + C_6 \cdot C_0$$
$$= 1 \cdot 132 + 1 \cdot 42 + 2 \cdot 14 + 5 \cdot 5 + 14 \cdot 2 + 42 \cdot 1 + 132 \cdot 1 = 429.$$

But remember that Iupuius wants to avoid configurations where one arch covers all others. Then we have to remove $C_6$ invalid configurations to get a grand total of $429 - 132 = 297$.

For the case of 10 arches we can compute $C_{10} - C_9 = 16796 - 4862 = 11934$. 
2. **Problem 2**

It is impossible to find an equilateral triangle all of whose vertices have integer coordinates, but one can get really close. For instance, the points (0, 0), (3, 11), and (11, 3) form an isosceles triangle with sides $\sqrt{130}$, $\sqrt{130}$, and $\sqrt{128}$.

(a) Show that there are infinitely many positive integers $n$ for which there are pairs of positive integers $r$ and $s$ such that the points $(0, 0)$, $(s, r)$, and $(r, s)$ form a triangle with sides $\sqrt{n}$, $\sqrt{n}$, and $\sqrt{n+1}$.

(b) Show that there are no positive integer pairs $r, s$ such that the points $(0, 0)$, $(s, r)$, and $(r, s)$ form a triangle with sides $\sqrt{n}$, $\sqrt{n}$, and $\sqrt{n-1}$.

Let’s look at the second part first. We want to prove that there are no integer solutions to

$$2(r - s)^2 - (r^2 + s^2) = -1.$$  

This is easy. Simplifying and taking both sides mod 4, we get $r^2 + s^2 = 3$. However, integer squares can only be 0 or 1 mod 4, so the only possible sums are 0, 1, and 2.

The first part is harder but there are several approaches. First, one might discover by trial and error that the following pairs work: $(1, 0)$, $(4, 1)$, $(15, 4)$, $(56, 15)$. At this point, it’s not too hard to discover that the sequence 0, 1, 4, 15, 56, … satisfies the recurrence relationship

$$f(n) = 4f(n - 1) - f(n - 2).$$

Using mathematical induction: Base case: $2(1 - 0)^2 = (1^2 + 0^2) + 1$ or, if you prefer, $2(4 - 1)^2 = (4^2 + 1^2) + 1$. Inductive step: Does

$$2(f(n) - f(n - 1))^2 = (f(n)^2 + f(n - 1)^2) + 1,$$

assuming that

$$2(f(n - 1) - f(n - 2))^2 = (f(n - 1)^2 + f(n - 2)^2) + 1?$$

Or, more simply, does

$$f(n)^2 - 4f(n)f(n - 1) + f(n - 1)^2 = 1$$

assuming that

$$f(n - 1)^2 - 4f(n - 1)f(n - 2) + f(n - 2)^2 = 1$$

Using the recursion, we restate $f(n)$ as $4f(n - 1) - f(n - 2)$ to get: Does

$$16f(n - 1)^2 - 8f(n - 1)f(n - 2) + f(n - 2)^2 - 4(4f(n - 1)^2 - f(n - 1)f(n - 2)) + f(n - 1)^2 = 1$$

assuming that

$$f(n - 1)^2 - 4f(n - 1)f(n - 2) + f(n - 2)^2 = 1?$$
Simplifying the expression, we find that the terms are exactly what we want -

\[ f(n - 1)^2 - 4f(n - 1)f(n - 2) + f(n - 2)^2, \]

which was assumed to equal 1.

This recursion can also be discovered/generated by a trigonometric argument: If \((0, 0), (s, r), \text{ and } (r, s)\) are the vertices of an equilateral triangle, then \(r/s = \tan(15^\circ) \text{ or } \tan(75^\circ)\); \(\tan(15^\circ)\) and \(\tan(75^\circ)\) are \(2 \pm \sqrt{3}\), respectively, and these are the roots of the quadratic equation \(x^2 - 4x + 1 = 0\). So if we assume that we are looking for a recursion of some sort which produces consecutive ratios near \(2 + \sqrt{3}\), we might well start by looking for a base case and expecting that use of the recursive equation \(f(n) - 4f(n - 1) + f(n - 2) = 0\).
Find all functions $f$ that take integer input and give integer output, and which satisfy the formula $f(a + b + f(b)) = f(a) + 2b$.

Claim 1: $f(0) = 0$.
By taking $a = b = 0$, we get that $f(0) = f(f(0))$. By taking $b = 0$, we see that $f(a + f(0)) = f(a)$ for any $a$, which implies that $f(f(0)) = f(2f(0))$. The first equality then yields that $f(2f(0)) = f(0 + f(0) + f(f(0)))$, which must be equal to $f(0) + 2f(0) = 3f(0)$. That is, $f(0) = 3f(0)$ and the claim follows.

Claim 2: if $f(a) = a$, then $f(na) = na$ for all integer $n$.
According to Claim 1, it holds that $2a = f(0 + a + f(a)) = f(2a)$. This establishes the base cases for $n = 1, 2$. Assume now that the claim is true for all $i = 1, 2, \ldots, k$. Then

$$f((k + 1)a) = f((k - 1)a + a + f(a)) = f((k - 1)a) + 2a = (k + 1)a.$$ 

Thus, the claim holds for all positive integers $n$ by the Principle of Mathematical Induction. Since $a = f(a) = f(-a + a + f(a)) = f(-a) + 2a$, it holds that $f(-a) = -a$. Therefore, the previous reasoning shows that $f(-na) = -na$ for all positive integers $n$, which finishes the proof of the claim.

Claim 3: $f(n) = nf(1)$ for all non-negative integers $n$.
It holds that $f(1 + f(1)) = f(0 + 1 + f(1)) = 2$. Then

$$f(n + 2) = f(n + 1 + f(1)) = f(n - 2 + 1 + f(1) + f(1 + f(1))) = f(n - 2) + 2(1 + f(1)),$$

which gives is $f(n) = f(n - 2) + 2f(1)$. Since $f(1) = 1f(1)$ and $f(0) = 0f(1)$ by Claim 1, Claim 3 follows from the Principle of Mathematical Induction.

Claim 4: either $f(n) = n$ for all integers $n$ or $f(n) = -2n$ for all integers $n$.
Let $A = 2 + f(1)$. Then $f(|A|) = |A|$. Indeed,

$$f(A) = f(1 + 1 + f(1)) = f(1) + 2 = A \quad \text{and} \quad f(-A) = -A$$

where the second equality follows from Claim 2. By Claim 3, $f(|A|) = |A|f(1)$. Hence, if $A \neq 0$, then $f(1) = 1$ and respectively $f(n) = n$ for all integers $n$ by Claim 3. Otherwise, $A = 0$, which means that $f(1) = -2$. Consequently, $f(n) = -2n$ for all non-negative integers $n$ by Claim 3. Finally, notice that

$$f(-n) = f(n + f(n)) = f(0 + n + f(n)) = 2n$$

for all positive integers $n$, which finishes the proof of Claim 4.
4. PROBLEM 4

You are given an arbitrary triangle $ABC$ inscribed in a circle. Show how to construct, using only a compass and straight edge, a triangle $DEF$ that is similar to $ABC$, with corresponding sides parallel, and such that $D$ lies on the segment $BC$, while $E$ and $F$ lie on the circular arc from $B$ to $C$ that does not contain $A$.

This problem would be a lot easier if it specified that $ABC$ was isosceles, because then we’d have a specific point on $BC$ to work from and known angles to use to determine the other two points. Strangely thinking about the problem we don’t have makes it easier to solve the problem we do have, because we can use exactly this technique to construct a rectangle with $BC$ as a base, $A$ on the opposite side, and then construct a similar rectangle meeting the constraints we want. The reason this is helpful is that a similar triangle to $ABC$ can be inscribed in a similar rectangle to the one in which $ABC$ will shortly be inscribed.

For the purposes of this solution, I will take basic constructions as given.

First, construct a line parallel to $BC$ through $A$. Next, construct perpendiculars to $BC$ at $B$ and $C$ so that we have the rectangle in question. Call the rectangle $C'B'B'C$ (with $A$ between $C'$ and $B'$). Then construct the midpoints of $BC$ and $B'C'$. Call them $G$ and $G'$, respectively, see Figure 1.

![Figure 1](image)

Now we create a similar rectangle to $C'B'B'C$ such that two points lie on the circular arc between $B$ and $C$ and the other two are between $B$ and $C$. Construct line segments $G'B$ and $G'C$, and then construct lines through $G$ parallel to these segments. The intersection with these two lines emanating from point $G$ ($E$ and $F$) are the base of our similar rectangle (and, hence, triangle), see Figure 2.
Getting the point corresponding to the vertex $A$ can be done in many ways but by far the simplest is to find the intersection of $BE$ and $CF$. Call this point $A'$. Then the point at which $AA'$ intersects $BC$ is $D$, see Figure 3.
2015 Team Problem Solution

Problem: There are many configurations of six squares that can be folded to form a cube; let us call them patterns. You are given a 10×10 sheet of paper, and tasked with cutting patterns from it to make as many 1×1×1 cubes as possible.

Solution: I was easily able to work out how to get 14 in my head, using just two cube-forms which could each tile rows very densely:

![Cube-form 1]

It was only late in the game when I discovered I could fit in a 15th cube-form as follows,

![Cube-form 2]

by fitting in a capital T in the largest gap (one of two choices for that gap, actually).

There are, as it happens, 11 different hexomino cube-forms (this is easy to Google):

![Hexomino cube-forms]

And of the two team solutions which presented 15 cube-forms, each used a much less limited palette, the first using five different cube-forms in its rendering and the second using seven. (Their solutions are presented below.) Well done!
As an aside, because each of the 35 hexominoes (including the 11 cube-forms) can be used to tile the plane, as n gets large the number of cube-forms of ANY SINGLE variety that can be cut from an nxn sheet is asymptotically equal to n^2/6.