1. Suppose that $X_1, \ldots, X_n$ is a random sample from the probability density function

$$f(x|\theta) = \frac{r x^{r-1}}{\theta} e^{-x^r/\theta}, \quad 0 < x < \infty,$$

where $r > 1$ is a known constant and $\theta > 0$ is an unknown parameter.

(a) Find an estimator of $\theta$ by the method of moments.
(b) Find the MLE of $\theta$.
(c) Are the estimators in (a) and (b) consistent? (Show why or why not.)
(d) Which estimator, (a) or (b), would you favor using and why?
(e) Based on the MLE for $\theta$, find an unbiased estimator of $\theta$.
(f) Based on the asymptotic distribution of the MLE for $\theta$, construct a 95% confidence interval for $\theta$.

2. Let $X_1, \ldots, X_n$ be a random sample from a Poisson($\lambda$) distribution, where $\lambda > 0$ is the mean parameter.

(a) Find the uniformly minimum variance unbiased estimator (UMVUE) $\tilde{\psi}_n$ and the maximum likelihood estimator (MLE) $\hat{\psi}_n$ of $\psi = e^{-\lambda}$.
(b) Compare the UMVUE and the MLE of $\psi$. Are the two similar or quite different? Explain.
(c) Show that, for any $r \in [0, 0.5)$, the UMVUE satisfies $n^r(\tilde{\psi}_n - \psi) \rightarrow 0$ in probability as $n \rightarrow \infty$.
(d) Derive the UMVUE $\tilde{\eta}_n$ of $\eta = P\{X_i \leq 1\}$. Is it also a consistent estimator of $\eta$?

3. Let $X_1, \ldots, X_n$ be IID beta($\nu^{-1}, 1$), where $0 < \nu < \infty$ is an unknown parameter.

(a) Derive the distribution of $Y_i = -\ln X_i$, for $1 \leq i \leq n$.
(b) What is the maximum likelihood estimator (MLE) of $\nu$? Is the MLE of $\nu$ also a complete sufficient statistic for $\nu$?
(c) What is the best 5% level critical region for testing $H_0 : \nu \leq 10$ versus $H_1 : \nu > 10$? In what sense is it the best?
(d) How big a sample is needed so that the above test will attain a 10% probability of type II error at $\nu = 5.6$?
(e) Develop a sequential probability ratio test for testing $H_0 : \nu = 5$ versus $H_2 : \nu = 5.6$ that will attain a 5% probability of type I error and a 10% probability of type II error.

4. A photocopy machine is fitted with a counter that records the number of copies made between successive breakdowns of the machine. After the repair person fixes the machine she resets the counter to zero. Below are the data on number of copies made between successive breakdowns.

7638, 1037, 5982, 20292, 20132, 13110, 4438, 4075, 12517, 14869, 2571, 32891

The summary statistics are: $n = 12, \bar{x} = 11629.3, s_x = 9360.1$.

We are interested in finding a 95% confidence interval for $\mu$, the average number of copies made before the machine breaks down. First, decide whether the data follow an exponential distribution. If it does, find a parametric confidence interval; and if it does not, find a non-parametric confidence interval for $\mu$. 