Problem 1. Suppose that $A \subset [0, 1]$ is a measurable set. Prove that the set

$$B = \cos A \equiv \{ \cos x, \ x \in A \}$$

is measurable and $mB \leq 0.85mA$, where $m$ is the Lebesgue measure.

*Hint.* Use that $\sin 1 = 0.841\ldots$

Problem 2. Prove that for any integrable function $f$ on $[a, b]$,

$$\lim_{n \to \infty} \int_a^b f(x) \sin^2(nx) dx = \frac{1}{2} \int_a^b f(x) dx.$$

Problem 3. Prove that there is no positive integrable function $f(x)$ on $[0, 1]$ such that

$$\lim_{n \to \infty} \int_0^1 [f(x)]^n dx = 2.$$

Problem 4. Let $A$ be a measurable set on $[0, 1]$. A point $x \in [0, 1]$ is called a *density point* of $A$ if

$$\lim_{\varepsilon \to 0} \frac{m(A \cap [x, x+\varepsilon])}{|\varepsilon|} = 1,$$

where $m$ stands for the Lebesgue measure. Prove that almost all points of the set $A$ are its density points.

*Hint.* Apply the theorem on the differentiation of an integral.

Problem 5. Let $f \in L^p(X)$, where $(X, \mathcal{A}, \mu)$ is a finite measure space, $\mu(X) < \infty$, and let $p > r \geq 1$. Prove that then $f \in L^r(X)$ and

$$\|f\|_r \leq (\mu X)^{\frac{1}{p} - \frac{1}{r}} \|f\|_p.$$

Problem 6. Prove that there exists the limit,

$$\lim_{n \to \infty} \int_0^1 \frac{e^{-x} \cos x}{n x^2 + \frac{1}{n}} dx,$$

and find it.

*Hint.* Make a change of variable.