§8. APPLICATIONS (Marshall Cohen)

We study closed embeddings $f: X \xrightarrow{\cong} f(X) \subset \mathbb{R}^n$ and show that $\tilde{H}_*(\mathbb{R}^n - f(X))$ is independent of f.! From this we deduce the Jordan-Brouwer separation theorem and invariance of domain

(8.1) Geometric prologue: the Klee trick

Suppose $X \subset \mathbb{R}^n \times \{0_m\} \subset \mathbb{R}^{n+m}$, $Y \subset \{0_n\} \times \mathbb{R}^m \subset \mathbb{R}^{n+m}$

$h: X \xrightarrow{\cong} Y$

Then $\exists$ a homeomorphism $H: \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ such that $H|X = h$

Proof. There is a continuous function $\bar{h}: \mathbb{R}^n \times \{0_m\} \to \{0_n\} \times \mathbb{R}^m$ such that $\bar{h}|X = h$

Reason: Let $h = (h_1, \dots, h_m)$, $h_i: X \to \mathbb{R}^1$, $X \subset \mathbb{R}^n$ = normal space.

By Tietse Extension Theorem (Munkres p. 212), h_i extends to $\bar{h}_i: \mathbb{R}^n \to \mathbb{R}^1$. Let $\bar{h} = (\bar{h}_1, \dots, \bar{h}_m)$

Let $\bar{H}: \mathbb{R}^{n+m} \xrightarrow{\cong} \mathbb{R}^{n+m}$ by $\bar{H}(x,y) = (x, y + \bar{h}(x))$

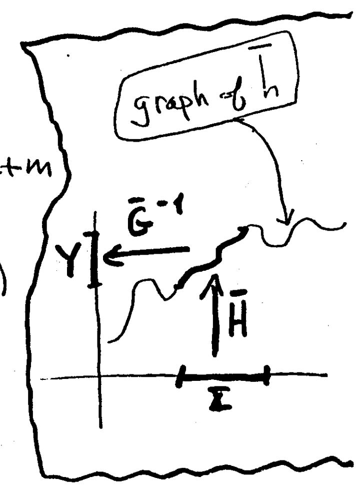
Similarly: If $g = h^{-1}: Y \to X$, $\exists\, \bar{g}: \mathbb{R}^m \to \mathbb{R}^n$ such that $\bar{g}|Y = g$. Let $\bar{G}: \mathbb{R}^{n+m} \xrightarrow{\cong} \mathbb{R}^{n+m}$ by

$$\bar{G}(x,y) = (x + \bar{g}(y), y)$$

Let $H = \bar{G}^{-1} \circ \bar{H}: \mathbb{R}^{n+m} \xrightarrow{\cong} \mathbb{R}^{n+m}$

Then: $\left.\begin{matrix} x \in X \\ h(x) \in Y \end{matrix}\right\} \Rightarrow H(x,0) = \bar{G}^{-1} \circ \bar{H}(x,0)$

$$= \bar{G}^{-1}(x, 0 + \bar{h}(x))$$
$$= (x - \bar{g}\bar{h}(x), \bar{h}(x))$$
$$= (0, \bar{h}(x)) \underset{x \in X}{=} (0, h(x))$$

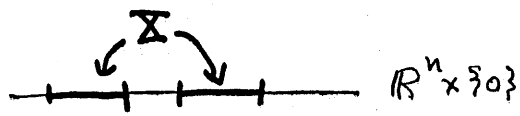


$\square$

(8.2) If $X \subset \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$, $X \neq \mathbb{R}^n$

then $\tilde{H}_k(\mathbb{R}^n - X) \cong \tilde{H}_{k+1}(\mathbb{R}^{n+1} - X)$ for all k

Proof:



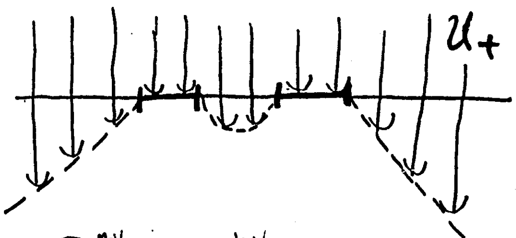
Consider the homeomorphisms $h_+, h_- : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$

given by

$$\left.\begin{aligned} h_+(x,t) &= (x,\ t + d(x,X)) \\ h_-(x,t) &= (x,\ t - d(x,X)) \end{aligned}\right\} \text{inverses of each other}$$

Let $U_+ = h_-(\mathbb{R}^{n+1}_+)$

$U_- = h_+(\mathbb{R}^{n+1}_-)$



Then $U_\pm \approx \mathbb{R}^{n+1}$, $U_\pm \subset \mathbb{R}^{n+1}$, $\mathbb{R}^{n+1} - X = U_+ \cup U_-$

$$U_+ \cap U_- \cong \mathbb{R}^n - X \times (-1,1) \searrow \mathbb{R}^{n+1} - X \times \{0\}$$

$$\therefore \to \underbrace{\tilde{H}_{k+1}(U_+)}_{0} \oplus \underbrace{\tilde{H}_{k+1}(U_-)}_{0} \to \tilde{H}_{k+1}(\mathbb{R}^{n+1} - X) \xrightarrow[\cong]{\partial} \tilde{H}_k(\mathbb{R}^n - X) \to 0 \oplus 0$$

using homotopy axiom

□

(8.3) **Theorem**: If $f : X \to f(X) \subset \mathbb{R}^n$

non-surjective closed, $g : X \to g(X) \subset \mathbb{R}^n$

are two embeddings of a space X into $\mathbb{R}^n$

then $\tilde{H}_k(\mathbb{R}^n - f(X)) \cong \tilde{H}_k(\mathbb{R}^n - g(X))$ for all $k \in \mathbb{Z}$

X, Y homeomorphic ($X = f(X)$), $Y = g(X)$

Proof: $\tilde{H}_k(\mathbb{R}^n - X) \overset{(8.2)}{\cong} \tilde{H}_{k+1}(\mathbb{R}^{n+1} - (X \times \{0\}))$

$\cong \tilde{H}_{k+2}(\mathbb{R}^{n+2} - (X \times \{0_2\}))$

$\cdots$

$$\left.\begin{aligned} &\cong \tilde{H}_{k+n}(\mathbb{R}^{n+n} - (X \times \{0_n\})) \\ &\overset{(8.1)}{\cong} \tilde{H}_{k+n}(\mathbb{R}^{2n} - (\{0_n\} \times Y)) \end{aligned}\right\} \text{homeomorphic spaces!}$$

$\cdots$

$\cong \tilde{H}_{k+1}(\mathbb{R}^{n+1} - (\{0\} \times Y))$

$\cong \tilde{H}_k(\mathbb{R}^n - Y)$

$\mathbb{R}^{n+1} - X = U_+ \cup U_-$: Say $(x,t) \in \mathbb{R}^{n+1} - X$. Then if $t=0$,

(8.3)* If $f,g: X \to S^n$ are non-surjective embeddings of a non-empty compact space into S^n then $\tilde{H}_k(S^n - f(X)) \cong \tilde{H}_k(S^n - g(X))$ for all $k \in \mathbb{Z}$

Proof: WLG $\exists\, a \in X$ such that $f(a) = g(a) = x_0 \in f(X) \cap g(X)$

$f(X) - \{x_0\} \subset S^n - \{x_0\} \cong \mathbb{R}^n$; $g(X) - \{x_0\} \subset S^n - \{x_0\} \cong \mathbb{R}^n$.

Then
$$\begin{aligned}\tilde{H}_k(S^n - f(X)) &\cong \tilde{H}_k((S^n - \{x_0\}) - (f(X) - \{x_0\})) \\ &\cong \tilde{H}_k((S^n - \{x_0\}) - (g(X) - \{x_0\})) \\ &\cong \tilde{H}_k(S^n - g(X)). \quad \square\end{aligned}$$

(8.4) If $k > n$ then a) $S^{k-1} \not\hookrightarrow \mathbb{R}^n$ and b) $\mathbb{R}^k \not\hookrightarrow \mathbb{R}^n$

Proof: Clearly a) $\Rightarrow$ b). To see a): $[S^{k-1} \cong \Sigma^{k-1} \subset \mathbb{R}^n]$ $(k-1 \geq n)$ $\Rightarrow [\underset{\mathbb{R}^n \times \{0_{k-n}\}}{\mathbb{R}^k - \Sigma^{k-1}}$ is connected while $\tilde{H}_0(\mathbb{R}^k - S^{k-1}) = \mathbb{Z}]$, contradicting (8.3). $\square$

Σ^{k-1} is compact — standard embedding

(8.5) If $S^k \cong \Sigma^k \subset S^n$ then $\tilde{H}_j(S^n - \Sigma^k) \cong \begin{cases} \mathbb{Z} & j = n-k-1 \\ 0 & \text{otherwise} \end{cases}$

Reason:
$$\begin{aligned}\tilde{H}_j(S^n - \Sigma^k) &\cong \tilde{H}_j(S^n - S^k) \cong \tilde{H}_j(S^k * S^{n-k-1} - S^k) \\ &\cong \tilde{H}_j(S^{n-k-1} \times (\text{open cone on } S^k)) \cong \tilde{H}_j(S^{n-k-1}). \quad \square\end{aligned}$$

(8.6) Jordan-Brouwer Separation Theorem

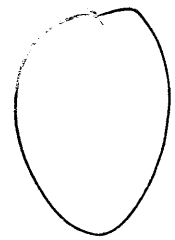
If $S^{n-1} \cong \Sigma^{n-1} \subset S^n$ (or $\mathbb{R}^n$)

then $S^n - \Sigma^{n-1} = U \sqcup V$ where

- U, V are open connected sets
- $\Sigma^{n-1} = \text{Bdy}\, U = \text{Bdy}\, V$

Proof: 1) $S^n - \Sigma^{n-1}$ is an open set. Since S^n is locally path connected, each component of $S^n - \Sigma^{n-1}$ is open. $\tilde{H}_0(S^n - \Sigma^{n-1}) \cong \mathbb{Z} \Rightarrow \exists$ two components, the U and V we seek

$n = 2 \quad k > 3$

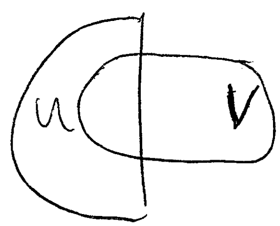


$X_0 \in B_\varepsilon$

path from U to V

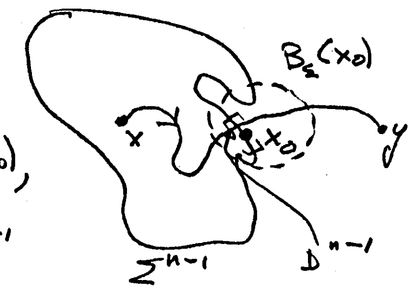
passing through $B_\varepsilon \cap \Sigma^{n-1}$

So pts.



Jordan-Brouwer proof continued

2) We show that $x_0 \in \Sigma^{n-1}$, $\varepsilon > 0 \Rightarrow B_\varepsilon(x_0) \cap U \neq \emptyset$ and $B_\varepsilon(x_0) \cap V \neq \emptyset$



Let D^{n-1} be an $(n-1)$-disk in Σ^{n-1} with $x_0 \in \mathring{D}^{n-1} \subset B_\varepsilon(x_0)$, $\mathring{D}^{n-1}$ = image of a round disk in S^{n-1}

Then $\tilde{H}_0(S^n - (\underbrace{\Sigma^{n-1} - \mathring{D}^{n-1}}_{\cong B^{n-1}})) = 0$

Let $x \in U$ an $y \in V$. $\exists$ a path in complement in S^n of $\Sigma^{n-1} - \mathring{D}^{n-1}$ connecting x, y. Since $U \amalg V$ is not connected, this path must meet Σ^{n-1} — hence must meet $\mathring{D}^{n-1}$ in some pt x_1 which is the last point on the path in $\bar{U}$. Hence $x_1 \in \mathring{D}^{n-1}$ is a limit of pts in U. Similarly $\exists$ $x_2 \in \mathring{D}^{n-1}$ in $\bar{V} - V$.

Since $d(x_0, x_i) < \underset{\uparrow \text{ arbitrary}}{\varepsilon}$, we see that x_0 is $\in$ Bdy $U \cap$ Bdy V $\square$